SELF-DUAL METRICS AND TWENTY-EIGHT BITANGENTS

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Abstract. We study self-dual metrics on $3\mathbf{CP}^2$ of positive scalar curvature admitting a non-zero Killing field, but which are not conformally isometric to LeBrun's metrics. Firstly, we determine defining equations of the twistor spaces of such self-dual metrics. Next we prove that conversely, the complex threefolds defined by the equations always become twistor spaces of self-dual metrics on $3\mathbf{CP}^2$ of the above kind. As a corollary, we determine a global structure of the moduli spaces of these self-dual metrics; namely we show that the moduli space is naturally identified with an orbifold $\mathbf{R}^3/G$, where G is an involution of $\mathbf{R}^3$ having two-dimensional fixed locus. Combined with works of LeBrun, this settles a moduli problem of self-dual metrics on $3\mathbf{CP}^2$ of positive scalar curvature admitting a non-trivial Killing field. In particular, it is shown that any two self-dual metrics on $3\mathbf{CP}^3$ of positive scalar curvature admitting a non-zero Killing field can be connected by deformation keeping the self-duality. In our proof, a key role is played by a classical result in algebraic geometry that a smooth plane quartic always possesses twenty-eight bitangents.

Contents

1. Introduction

A Riemannian metric on an oriented four-manifold is called self-dual if the anti-self-dual part of the Weyl conformal curvature of the metric identically vanishes. Basic examples are provided by the round metric on the four-sphere and the Fubini-Study metric on the

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complex projective plane. In general, one can expect that if two four-manifolds admit self-dual metrics respectively, then their connected sum will also admit a self-dual metric. In fact, Y.S. Poon [14] constructed explicit examples of self-dual metrics on $2\mathbf{CP}^2$, the connected sum of two complex projective planes. He further showed that on $2\mathbf{CP}^2$ there are no self-dual metrics other than his metrics, under assumption of the positivity of the scalar curvature. Later, C. LeBrun [10] and D. Joyce [7] respectively constructed a family of self-dual metrics of positive scalar curvature on $n\mathbf{CP}^2$ for any $n \geq 1$. These are called LeBrun metrics and Joyce metrics respectively, and have nice characterizations by the (conformal) isometry group. Namely, A. Fujiki [2] proved that if a self-dual metric on $n\mathbf{CP}^2$ has an effective $U(1) \times U(1)$-isometry, then it must be a Joyce metric. LeBrun [11] showed that if a self-dual metric on $n\mathbf{CP}^2$ has a non-trivial semi-free $U(1)$-isometry, then the metric must be a LeBrun metric. Here, a $U(1)$-action on a manifold M is called semi-free if the isotropy group is $U(1)$ or identity only, at every point of M. By his construction, LeBrun metrics form a connected family for each n.

Now drop the assumption of the semi-freeness of $U(1)$-isometry. A work of Pedersen-Poon [13] suggests that there should exist self-dual metrics on $n\mathbf{CP}^2$, $n \geq 3$, admitting non-semifree $U(1)$-isometry. In fact, this is shown to be true by the author [5, 6]. However, it seems to be quite difficult to classify *all* self-dual metrics on $n\mathbf{CP}^2$ with $U(1)$-isometry, for arbitrary n. The purpose of the present paper is to settle this problem for $3\mathbf{CP}^2$. Namely, we show the following:

Theorem 1.1. *Let $\mathcal{M}$ be the moduli space of conformal classes of self-dual metrics on $3\mathbf{CP}^2$ satisfying the following conditions: (i) having a positive scalar curvature, (ii) admitting a non-zero Killing field, or equivalently, a non-trivial $U(1)$-isometry, (iii) being not conformal to LeBrun metrics. Then $\mathcal{M}$ is non-empty and is naturally identified with an orbifold $\mathbf{R}^3/G$, where G is an involution on $\mathbf{R}^3$ having two-dimensional fixed locus.*

Thus a global structure of the moduli space is determined. In particular, it is connected as in the LeBrun's case. We note that the claims of the theorem involve the uniqueness of the $U(1)$-action in (ii). The $U(1)$-action is inequivalent to LeBrun's semi-free action.

Next we mention a relationship between our self-dual metrics and LeBrun's self-dual metric. We showed in [5] that some of self-dual metrics on $3\mathbf{CP}^2$ satisfying three conditions in Theorem 1.1 can be obtained as a small deformation of Joyce metric with torus symmetry. Because Joyce metrics on $3\mathbf{CP}^2$ fall into a subfamily of LeBrun metrics, Theorem 1.1 implies the following

Corollary 1.2. *Any two self-dual metrics on $3\mathbf{CP}^2$ satisfying (i) and (ii) in Theorem 1.1 can be connected by deformation keeping the self-duality.*

Alternatively, one can also state that both of our self-dual metrics and LeBrun metrics have a Joyce metric (or LeBrun metric with torus action) as a limit. Because the Killing field (or associated $U(1)$-action) of ours and LeBrun's are different, one must exchange the Killing filed when passing through a LeBrun metric with torus symmetry.

Now we explain why the involution G appears in Theorem 1.1. In the course of our proof of Theorem 1.1, we construct a family $\tilde{\mathcal{M}}$ of self-dual metrics on $3\mathbf{CP}^2$ parameterized by $\mathbf{R}^3$, which contains all metrics satisfying conditions (i)–(iii). But it will turned out that the family does not effectively parameterize the metrics; nemaly we see that an involution G acts on this parameter space $\mathbf{R}^3$ in such a way that two points exchanged by the involution represent the same conformal class. Fixed points of G, which will turn out to be two-dimensional, represent self-dual metrics having a non-trivial isometric involution, where non-trivial means that the involution does not belong to $U(1)$.

Our proof of Theorem 1.1 is based on the twistor theory. Namely, by using so called the Penrose correspondence between self-dual metrics and complex threefolds called twistor

spaces, we translate the problem into that of complex algebraic geometry. Rather surprisingly, this translation leads us to an unexpected connection between self-dual metrics and bitangents of a plane quartic curve which is a classical but fruitful topic in algebraic geometry. It is this connection which enable us to find *arbitrary* twistor lines in certain complex threefolds, which is equivalent to solving the self-duality equation for Riemannian metrics on a f-manifold[1].

In Section 2 we first determine defining equations of the twistor spaces of self-dual metrics on $3\mathbf{CP}^2$ satisfying the above three conditions in Theorem 1.1. More concretely, we prove that the twistor spaces have a structure of generically two-to-one covering branched along a quartic

$$(y_2y_3 + Q(y_0, y_1))^2 - y_0y_1(y_0 + y_1)(y_0 - ay_1) = 0, \tag{1}$$

where $Q(y_0, y_1)$ is a quadratic form of y_0 and y_1 with real coefficients, and a is a positive real number (Proposition 2.1). Further, we show that Q and a must satisfy certain inequality, which we call Condition (A) (Proposition 2.6).

Most of the rest of the paper is devoted to proving that, conversely, a complex threefold Z having a structure of generically two-to-one covering branched along the quartic (1) has a structure of a twistor space of $3\mathbf{CP}^2$. Since the quartic surface (1) always has isolated singularities, the double covering has isolated singularities over there. For this space, we prove the following theorem completing inversion construction of twistor spaces:

Theorem 1.3. *For any quartic surfaces B of the form (1) satisfying the condition (A), there exists a resolution of the double covering of $\mathbf{CP}^3$ branched along B, such that the resulting smooth threefold is a twistor space of $3\mathbf{CP}^2$.*

In fact, the main result of this paper (Theorem 10.1) implies more. Namely, we prove that there exist precisely two (small) resolutions of the double cover which are actually twistor spaces, and that these two spaces are isomorphic as twistor spaces so that they determine the same self-dual structure.

It is relatively easy to derive Theorem 1.1 (= Theorem 10.4) from Theorem 10.1. As is already mentioned, our strategy for proving this result is to find the family of twistor lines in Z; namely a real four-dimensional family of real smooth rational curves which foliates the whole of the threefold Z. In general, it is difficult (except a few simple cases) to find arbitrary twistor lines. However, in the present case this is possible basically because their images onto $\mathbf{CP}^3$ (by the covering morphism) must be, in general, conics with a very special property; namely they are conics touching the branch quartic surface (Proposition 3.2). We call this kind of conics in $\mathbf{CP}^3$ touching conics (following Hadan [3]). Roughly speaking, every touching conics are 'generated' from bitangents of the quartic, and the existence of twenty-eight bitangents guarantees the existence of twistor lines. This is how the self-dual metrics and twenty-eight bitangents are related.

Thus a large part of the problem of finding twistor lines in Z is reduced to finding touching conics of the quartic (1). Note that a conic in $\mathbf{CP}^3$ is always contained in a unique plane. Broadly speaking, our method of finding touching conics of the quartics consists of two parts: one is to find 'general' touching conics, and the other is to find 'degenerate' ones which are limits of the 'general' touching conics. Here, 'general' means that the plane on which the conic lies intersects the branch quartic (1) smoothly. It is possible to show that on this kind of planes there are 63 one-dimensional families of touching conics (Proposition 3.10; see also [3]). A family which can actually be the images of twistor lines must be unique and there are too many candidates of twistor lines, even if we take the reality into account. This seems to be main difficulty in proving that the threefolds are actually twistor spaces (cf. [9, 3]).

We are able to overcome this difficulty by considering 'degenerate' touching conics, whose planes intersect the quartic (1) non-smoothly. Thanks to a very special form of the equation

(1), it is possible to determine which plane intersects the quartic surface non-smoothly (Lemma 3.4). Then it is readily seen that the sections of the quartic surface by these planes are union of two irreducible conics. The point is that, for these planes, we can determine all families of ('degenerate') touching conics in very explicit form (Propositions 5.2, 5.4 and 5.5). These in particular show that the number of families of touching conics drastically decreases for these planes. Moreover, we can use this explicit descriptions of 'degenerate' touching conics to determine which family actually comes from twistor lines (Proposition 5.6). These are done in the latter half of Section 5 and Section 7 by actually investigating the inverse images (in Z) of 'degenerate' touching conics.

Thus we obtain candidates of twistor lines lying over planes intersecting the quartic surface non-smoothly. We can show that these candidates can be deformed in Z to give twistor lines whose images are 'general' touching conics, and that among the above 63 families of 'general' touching conics, there is a unique family which is obtained in this way (Proposition 9.1). This is how we obtain twistor lines whose images are (touching) conics. These form real four-dimensional family and cover an open subset of Z. However, this family does not cover the whole of Z. In order to cover the whole of Z, we need to consider another set of twistor lines whose images becomes *lines* in $\mathbf{CP}^3$ (cf. Proposition 3.2). These twistor lines are investigated in Section 8 and will be called 'twistor lines at infinity'. A conclusion is that the inverse images of real lines (in $\mathbf{CP}^3$) going through the unique real singular point of B (coming from Condition (A)) always contain a real twistor line as its irreducible component (Proposition 8.1). The parameter space of these twistor lines (at infinity) becomes a boundary for compactifying the parameter space of twistor lines whose images are touching conics.

In this way we get candidates of all twistor lines in Z. It remains to show that Z is actually foliated by these candidates. Namely, we have to show that different members of these candidates do not intersect, and that they cover the whole of Z. These will be proved in Proposition 9.3. The point of our proof is to consider the inverse images of arbitrary real lines, which are smooth elliptic curves in general, and sometimes degenerate into a cycle of rational curves. We show that these elliptic curves and their degenerations are foliated by circles which are intersection of the elliptic curve with the family of twistor lines lying over a real plane containing the original line. Finally we show that the parameter space of these twistor lines is actually $3\mathbf{CP}^2$, and complete the inversion construction as Theorem 10.1.

Finally, we give some comments concerning the quartic surfaces (1). The equation (1) itself appears in Y. Umezu's paper [16, p. 141], where she investigated normal quartic surfaces whose minimal resolutions are birational to elliptic ruled surfaces (which are by definition a $\mathbf{CP}^1$-bundle over an elliptic curve). In particular, she proved that if such a quartic surface has two simple elliptic singularities of type $\tilde{E}_7$, then one can take (1) as its defining equation. By counting the number of coefficients, these quartic surfaces form a four-dimensional family. Our Condition (A) imposes one more condition on the coefficients and the surface (1) has one more singularity (which is an ordinary double point). Consequently our moduli space becomes three-dimensional as in Theorem 1.1. Also it should be noted that the equation (1) is already appeared in the paper of Kreußler-Kurke [9], which they obtained while carefully classifying branch quartic surfaces for *general* twistor space (namely no assumptions on the Killing fields) of $3\mathbf{CP}^2$.

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Notations and conventions. Let X be a complex manifold and σ an anti-holomorphic involution on X. A subset $Y \subset X$ is said to be *real* if $\sigma(Y) = Y$. The set of σ-fixed points of

Y is denoted by Y^σ. A holomorphic line bundle F on X is said to be *real* if $\sigma^* F$ is isomorphic to $\overline{F}$, where $\overline{F}$ denotes the complex conjugation of F. A complete linear system $|F|$ is called real if F is a real line bundle, and then $|F|^\sigma \subset |F|$ denotes the subset consisting of real (i.e. σ-invariant) members. This is necessarily parametrized by the real projective space $\mathbf{RP}^{n-1}$, where n is the complex dimension of $H^0(F)$. Next assume that Z is a twistor space of a self-dual four-manifold (M, g). Let σ be the real structure which is an anti-holomorphic involution of Z. *A twistor line* is a fiber of the twistor fibration $Z \to M$. Next, suppose that Z is a three-dimensional complex manifold with an anti-holomorphic involution σ, for which we do not assume that Z is actually a twistor space. Then a non-singular complex submanifold $L \subset Z$ is called a *real line* if L is biholomorphic to the complex projective line and if its normal bundle in Z is isomorphic to $O(1)^{\oplus 2}$. Needless to say, twistor line is a real line. However, it is important to make a distinction of 'twistor lines' and 'real lines', because there is an example of twistor space possessing a real line which is not a twistor lines (as proved in Corollary 7.7).

2. Defining equations of the branch quartic surfaces and their singularities

Let g be a self-dual metric on $3\mathbf{CP}^2$ of positive scalar curvature, and assume that g is not conformally isometric to LeBrun metrics. Let Z be the twistor space of g, and denote by $(-1/2)K_Z$ the fundamental line bundle which is the canonical square root of the anticanonical line bundle of Z. These non-LeBrun twistor spaces of $3\mathbf{CP}^2$ are extensively studied in Kreußler and Kurke [9] and Poon [15], and it has been proved that the fundamental system (the complete linear system associated to the fundamental line bundle) is free and of three-dimensional, and induces a surjective morphism $\Phi : Z \to \mathbf{CP}^3$ which is generically two-to-one, and that the branch divisor B is a quartic surface with only isolated singularities. Furthermore, there is the following diagram:

$$\begin{array}{ccc} & Z & \\ \mu & \downarrow & \searrow \Phi \\ & Z_0 & \underset{\Phi_0}{\longrightarrow} \mathbf{CP}^3 \end{array} \tag{2}$$

where $\Phi_0 : Z_0 \to \mathbf{CP}^3$ denotes the double covering branched along B, and μ is *a small resolution* of the singularities of Z_0 over the singular points of B.

For generic non-LeBrun metric g, B has only ordinary double points [9, 15] and hence is birational to a K3 surface. As a consequence, one can deduce that Z does not admit a non-zero holomorphic vector field. However, the author showed in [5] and [6] that if B degenerates to have non-ADE singularities, then Z admits a non-zero holomorphic vector field, and that such a twistor space of $3\mathbf{CP}^2$ actually exists. Concerning the defining equation of the branch quartic B for such twistor spaces, we have the following proposition which is the starting point of our investigation.

Proposition 2.1. *Let g be a non-LeBrun self-dual metric on $\mathbf{CP}^2$ of positive scalar curvature, and assume the existence of a non-trivial Killing field. Let $\Phi : Z \to \mathbf{CP}^3$ and $B \subset \mathbf{CP}^3$ be as above. Then there exists a homogeneous coordinate $(y_0 : y_1 : y_2 : y_3)$ on $\mathbf{CP}^3$ fulfilling (i)-(iii) below:*
(i) a defining equation of B is given by

$$(y_2 y_3 + Q(y_0, y_1))^2 - y_0 y_1 (y_0 + y_1)(y_0 - a y_1) = 0, \tag{3}$$

where $Q(y_0, y_1)$ is a quadratic form of y_0 and y_1 with real coefficients, and a is a positive real number,
(ii) the naturally induced real structure on $\mathbf{CP}^3$ is given by

$$\sigma(y_0 : y_1 : y_2 : y_3) = (\overline{y}_0 : \overline{y}_1 : \overline{y}_3 : \overline{y}_2),$$

(iii) the naturally induced $U(1)$-action on $\mathbf{CP}^3$ is given by

$$(y_0 : y_1 : y_2 : y_3) \mapsto \left(y_0 : y_1 : e^{i\theta} y_2 : e^{-i\theta} y_3\right), \quad e^{i\theta} \in U(1).$$

Proof. If the fundamental system of Z is free, there are just four reducible members, all of which are real [15, 9]. We write $\Phi^{-1}(H_i) = D_i + \overline{D}_i$, $1 \le i \le 4$, where H_i is a real plane in $\mathbf{CP}^3$. The restrictions of Φ onto D_i and $\overline{D}_i$ are obviously birational morphisms onto H_i, so that, together with the fact that $(-1/2)K_Z \cdot L_i = 2$, it can be readily seen that $T_i := \Phi(L_i)$ is a conic contained in B. This implies that the restriction of B onto H_i is a conic of multiplicity two. Namely, T_i, $1 \le i \le 4$, is so called *a trope* of B.

A Killing field naturally gives rise to an isometric $U(1)$-action, which can be canonically lifted to a holomorphic $U(1)$-action on the twistor space. This action naturally goes down to $\mathbf{CP}^3$, and every subvarieties above are clearly preserved by these $U(1)$-actions. In particular, T_i is a $U(1)$-invariant conic on a $U(1)$-invariant plane H_i, where the $U(1)$-action is induced by the vector field. Since the twistor fibration is $U(1)$-equivariant and generically one-to-one on D_i, and since $\Phi|_{D_i} : D_i \to H_i$ is also $U(1)$-equivariant and birational, the $U(1)$-action on any H_i is non-trivial. Hence $U(1)$ acts non-trivially on T_i. For $j \ne i$, put $l_{ij} = H_i \cap H_j$, which is clearly a real $U(1)$-invariant line. Then $C_i \cap l_{ij}$ must be the two $U(1)$-fixed points on T_i, since it is real set and since there is no real point on T_i. This implies that l_{ij} is independent of the choice of $j \ne i$. So we write $l_{ij} = l_\infty$ and let P_∞ and $\overline{P}_\infty$ be the two fixed points of the $U(1)$-action on l_∞. Then H_i, $1 \le i \le 4$, must be real members of the real pencil of planes whose base locus is l_∞. Since l_∞ is a real line, we can choose real linear forms y_0 and y_1 such that $l_\infty = \{y_0 = y_1 = 0\}$. Further, since any of H_i is real, by applying a real projective transformation (with respect to (y_0, y_1)), we may assume that $\cup_{i=1}^4 H_i = \{y_0 y_1 (y_0 + y_1)(a y_0 - b y_1) = 0\}$, where $a, b \in \mathbf{R}$ with $a > 0$ and $b > 0$.

As seen above, every T_i goes through P_∞ and $\overline{P}_\infty$. Let l_i, $1 \le i \le 4$ be the tangent line of T_i at P_∞. Now we claim that l_i's are lying on the same plane. Let H be the plane containing l_1 and l_2. Then by using $l_1 \cap B = P_\infty = l_2 \cap B$, we can easily deduce that $B \cap H$ is a union of lines, all of which goes through P_∞. Suppose that l_3 is not contained in H. Then the line $H \cap H_3$ is not tangent to T_3, so there is an intersection point of $T_3 \cap H$ other than P_∞. Then the line $H \cap H_3$ is contained in B, because we have already seen that $B \cap H$ is a union of lines all passing through P_∞. This is a contradiction since $B \cap H_3 = 2T_3$. Similarly we have $l_4 \subset H$. Therefore $l_i \subset H$ for any i, as claimed. Because T_i's are real, the plane $\sigma(H)$ contains the tangent lines of T_i's at $\overline{P}_\infty$. Let y_3 be a linear form on $\mathbf{CP}^3$ defining H, and set $y_2 := \overline{\sigma^* y_3}$. Then $(y_0 : y_1 : y_2 : y_3)$ is a homogeneous coordinate on $\mathbf{CP}^3$. By our choice, we have $\sigma(y_0 : y_1 : y_2 : y_3) = (\overline{y}_0 : \overline{y}_1 : \overline{y}_3 : \overline{y}_2)$, $P_\infty = (0 : 0 : 0 : 1)$ and $\overline{P}_\infty = (0 : 0 : 1 : 0)$.

Because the planes $\{y_i = 0\}$ are $U(1)$-invariant, our $U(1)$-action can be linearized with respect to the homogeneous coordinate $(y_0 : y_1 : y_2 : y_3)$. Further, since H_i's are $U(1)$-invariant, the action can be written $(y_0 : y_1 : y_2 : y_3) \mapsto (y_0 : y_1 : e^{i\alpha\theta} y_2 : e^{i\beta\theta} y_3)$ for $e^{i\theta} \in U(1)$, where α and β are relatively prime integers. Moreover, since the conics T_i's are $U(1)$-invariant, we can suppose $\alpha = 1$, $\beta = -1$. Thus the $U(1)$-action can be written in the form (ii) of the proposition.

Let $F = F(y_0, y_1, y_2, y_3)$ be a defining equation of B. Since B is $U(1)$-invariant, monomials appeared in F must be in the ideal $(y_2 y_3, y_0^2, y_0 y_1, y_1^2)^2$. Moreover, F contains the monomial $y_2^2 y_3^2$, since otherwise the restriction onto $\{y_1 = 0\}$ would be the union of two different conics, which contradict to the fact that T_i is a trope. We assume that its coefficient is 1. Then F can be written in the form $(y_2 y_3 + Q(y_0, y_1))^2 - q(y_0, y_1)$, where $Q(y_0, y_1) \in (y_0, y_1)^2$ and $q(y_0, y_1) \in (y_0, y_1)^4$ are uniquely determined polynomials with real coefficients. Then it again follows from T_i being a trope that $q(y_0, y_1) = k y_0 y_1 (y_0 + y_1)(a y_0 - b y_1)$ for some constant $k \in \mathbf{R}^\times$. If k is negative, exchange y_0 and y_1. Then we get $(y_2 y_3 + Q(y_0, y_1))^2 -$

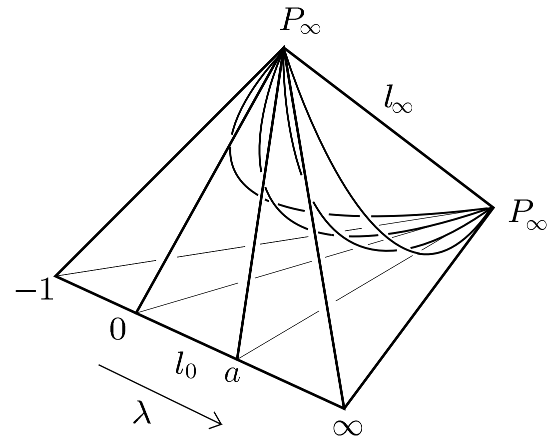

FIGURE 1. four planes containing the tropes of B

$ky_0y_1(y_0+y_1)(ay_0-by_1)$ where $k>0, a>0$ and $b>0$. This can be written

$$\left(\frac{1}{\sqrt[4]{ka}}y_2\cdot\frac{1}{\sqrt[4]{ka}}y_3+\frac{1}{\sqrt{ka}}Q(y_0,y_1)\right)^2-y_0y_1(y_0+y_1)\left(y_0-\frac{b}{a}y_0\right)=0.$$

Hence if we rewrite y_2 and y_3 for $(1/\sqrt[4]{ka})y_2$ and $(1/\sqrt[4]{ka})y_3$ respectively, and $Q(y_0,y_1)$ and a for $(1/\sqrt{ka})Q(y_0,y_1)$ and a respectively, we obtain (3) as an defining equation of B. Finally it is obvious that the forms of the real structure and the $U(1)$-action are invariant under the above change of coordinate. □

Note that by a result of Y. Umezu [16, p. 141, (1)], every normal surface in $\mathbf{CP}^3$ which is birational to an elliptic ruled surface and which has two simple elliptic singularities of type $\tilde{E}^7$ has (3) as its defining equation.

Next we study the singular locus of B.

Proposition 2.2. *Let B be a real quartic surface defined by the equation*

$$(y_2y_3+Q(y_0,y_1))^2-y_0y_1(y_0+y_1)(y_0-ay_1)=0$$

where $Q(y_0,y_1)$ is a real quadratic form of y_0 and y_1, and $a>0$. Let A be the set $\{(y_0:y_1:0:0)\,|\,(y_0:y_1)$ is a multiple root of the quartic equation $Q(y_0,y_1)^2-y_0y_1(y_0+y_1)(y_0-ay_1)=0\}$. (Here we think of this as an equation on $\mathbf{CP}^1=\{(y_0:y_1)\}$.) Then we have: (i) $\mathrm{Sing}(B)=\{P_\infty,\overline{P}_\infty\}\cup A$, where we put $P_\infty=(0:0:0:1)$, (ii) P_∞ and $\overline{P}_\infty$ are simple elliptic singularities of type $\tilde{E}_7$, and (iii) if $Q(y_0,y_1)\neq 0$, then $(y_0:y_1:0:0)\in A$ is an ordinary double point iff its multiplicity is two.

In particular, every singular point of B is isolated.

Proof. (i) First we show that $(\mathrm{Sing}B)\cap\{y_3\neq 0\}=\{P_\infty\}$, by calculating the Jacobian. Let $x_i=y_i/y_3$ $(0\le i\le 2)$ be affine coordinates on $y_3\neq 0$. Then the equation of B becomes $(x_2+Q(x_0,x_1))^2-x_0x_1(x_0+x_1)(x_0-ax_1)=0$. Differentiating with respect to x_2, we get $x_2+Q(x_0,x_1)=0$ so that we have $x_0x_1(x_0+x_1)(x_0-ax_1)=0$. Next differentiating with respect to x_0 and x_1 and then substituting $x_2+Q(x_0,x_1)=0$, we get $x_1(x_0+x_1)(x_0-ax_1)+x_0x_1(x_0-ax_1)+x_0x_1(x_0+x_1)=0$ and $x_0(x_0+x_1)(x_0-ax_1)+x_0x_1(x_0-ax_1)-ax_0x_1(x_0+x_1)=0$. From the former, we obtain that $x_0=0$ implies $x_1=0$. Then by $x_2+Q(x_0,x_1)=0$ we have $x_2=0$. Similar argument shows that if x_1, x_0+x_1 or x_0-ax_1 is zero, then $x_0=x_1=x_2=0$. Conversely, it is immediate to see that $(x_0,x_1,x_2)=(0,0,0)$ is a double point of B. Thus we get $(\mathrm{Sing}B)\cap\{y_3\neq 0\}=\{P_\infty\}$.

Because the given homogeneous polynomial is symmetric with respect to y_2 and y_3, we have $(\mathrm{Sing}B) \cap \{y_2 \neq 0\} = \{\overline{P}_\infty\}$.

Next we show that $(\mathrm{Sing}B) \cap \{y_2 = y_3 = 0\} = A$. We may suppose $y_1 \neq 0$. Putting $v_i = y_i/y_1$ for $i = 0, 2, 3$, the equation of B becomes $(v_2v_3 + Q(v_0, 1))^2 - f(v_0) = 0$, where we put $f(v_0) = v_0(v_0 + 1)(v_0 - a)$. Substituting $v_2 = v_3 = 0$, we get $Q(v_0, 1)^2 - f(v_0) = 0$. On the other hand, differentiating with respect to v_0 and substituting $v_2 = v_3 = 0$, we get $(Q(v_0, 1)^2 - f(v_0))' = 0$, where the prime denotes differential with respect to v_0. Thus, if $(v_0, v_2, v_3) = (\lambda_0, 0, 0)$ is a singular point of B, λ_0 is a multiple root of $Q(v_0, 1)^2 - f(v_0) = 0$. Conversely, it is easy to see that $(v_0, v_2, v_3) = (\lambda_0, 0, 0)$ is a singular point for such λ_0. Thus we get the claim of (i).

(ii) is obvious if one notes that we can use $(x_0, x_1, x_2 + Q(x_0, x_1))$ instead of (x_0, x_1, x_2) as a local coordinate around P_∞.

Finally we show (iii) by using the coordinate (v_0, v_2, v_3) above. Let λ_0 be a multiple root of $Q(v_0, 1)^2 - f(v_0) = 0$. Then our equation of B can be written $(v_2v_3 + 2Q(v_0, 1))v_2v_3 + g(v_0)(v_0 - \lambda_0)^2 = 0$, where $g(v_0)$ is a polynomial of degree two. Clearly λ_0 is a double root iff $g(\lambda_0) \neq 0$. Suppose $g(\lambda_0) \neq 0$ and define

$$\begin{aligned} w_1 &= \sqrt{g(v_0)} \cdot (v_0 - \lambda_0), \\ w_2 &= \sqrt{2Q(v_0, 1) + v_2v_3} \cdot v_2, \\ w_3 &= \sqrt{2Q(v_0, 1) + v_2v_3} \cdot v_3. \end{aligned} \tag{4}$$

Because $g(\lambda_0) \neq 0$ and $Q(\lambda_0, 1) \neq 0$, (w_1, w_2, w_3) is a local coordinate around $(\lambda_0, 0, 0)$. Then our equation of B becomes $w_2w_3 + w_1^2 = 0$. Thus the singularity is an ordinary double point. Conversely, if $g(\lambda_0) = 0$, it is immediate to see that our equation of B can be written of the form $w_2w_3 + w_1^3 = 0$ or $w_2w_3 + w_1^4 = 0$ depending on whether the multiplicity of λ_0 is three or four. This implies that $(\lambda_0, 0, 0)$ is not an ordinary double point. □

Proposition 2.3. *Let B be as in Proposition 2.2. Put $f(\lambda) = \lambda(\lambda + 1)(\lambda - a)$. Let $Z_0 \to \mathbf{CP}^3$ be the double covering branched along B. Then if Z_0 admits a small resolution $Z \to Z_0$ such that Z is a twistor space of $3\mathbf{CP}^2$, then $Q(\lambda, 1)^2 - f(\lambda) \geq 0$ for any $\lambda \in \mathbf{R}$ and the equality holds for a unique $\lambda_0 \in \mathbf{R}$. Further, in this case, the multiplicity of λ_0 is two.*

Note that it follows from this proposition that $f(\lambda_0) > 0$ holds, because we have $Q(\lambda_0, 1)^2 = f(\lambda_0)$ and $Q(\lambda_0, 1)$ is a real number which is non-zero because otherwise the restriction $B|_{H_{\lambda_0}}$ would be $y_0 - \lambda_0 y_1 = (y_2y_3)^2 = 0$ that yields another reducible fundamental divisor.

Proof. By results of Kreußler [8] and Kreußler-Kurke [9], we have $\sum(\mu(x) + c(x)) = 26$ for Z to be a twistor space of $3\mathbf{CP}^2$ for a topological reason, where $\mu(x)$ is the Milnor number of the singularity x of B and $c(x)$ is the number of irreducible components of a small resolution $Z \to Z_0$. Because elliptic singularity of type $\tilde{E}_7$ has $\mu = 9$ and $c = 3$, we get $\sum(\mu(x) + c(x)) = 2$ for other remaining singularities. This implies that there is only one singularity remaining, and that it must be an ordinary double point, which will be denoted by P_0. Therefore, by Proposition 2.2 (i), we have $A = \{P_0\}$. Namely, $Q^2 - f = 0$ has a unique multiple root λ_0. The multiplicity is two by Proposition 2.2 (iii). It is obvious from the uniqueness that this ordinary double point is real. Namely, λ_0 is real.

Next we show that other solutions of $Q(\lambda, 1)^2 - f(\lambda) = 0$ are not real. Assume $\lambda \in \mathbf{R}$, $\lambda \neq \lambda_0$ is a solution. Then by restricting B to the plane $y_0 = \lambda y_1$, we get $(y_2y_3 + Q(\lambda, 1)y_1^2)^2 - f(\lambda)y_1^4 = (y_2y_3)^2 + 2Q(\lambda, 1)y_2y_3y_1^2 + (Q(\lambda, 1)^2 - f(\lambda))y_1^4 = y_2y_3(y_2y_3 + 2Q(\lambda, 1)y_1^2)$ $(= 0)$. Therefore, the point $(\lambda : 1 : 0 : 0)$ is a real point of B. Since the multiplicity of the solution λ is one, Proposition 2.2 shows that this is a smooth point of B. This implies that Z has a real point, contradicting to the absence of real points on any twistor spaces. Hence the equation $Q(\lambda, 1)^2 - f(\lambda) = 0$ has no real solution other than λ_0. Because λ_0 is a solution

whose multiplicity is two, this implies that the polynomial $Q(\lambda,1)^2-f(\lambda)$ has constant sign on $\mathbf{R}\backslash\{\lambda_0\}$. This sign must be clearly positive. $\square$

To investigate the real locus of B, we need the following elementary

Lemma 2.4. *Let $C_\alpha=\{y_2y_3=\alpha y_1^2\}$, $\alpha\in\mathbf{R}$ be a real conic in $\mathbf{CP}^2$, where the real structure is given by $(y_1:y_2:y_3)\mapsto(\overline{y}_1:\overline{y}_3:\overline{y}_2)$. Then C_α has no real point iff $\alpha<0$.*

Proof. It is immediate to see that the real locus of C_α is

$$\left\{(1:v:\overline{v})\in\mathbf{CP}^2\,|\,|v|=\sqrt{\alpha}\right\}.$$

This is empty iff $\alpha<0$.

Proposition 2.5. *Let B be as in Proposition 2.2 and suppose that the inequality $Q(\lambda,1)^2-f(\lambda)\geq 0$ holds on $\mathbf{R}$ with the equality holding iff $\lambda=\lambda_0$ as in Proposition 2.3. Put $P_0:=(\lambda_0:1:0:0)$, which is clearly a real point of B. Then we have: (i) there is no real point on B other than P_0 iff the following condition is satisfied: if $f(\lambda)\geq 0$ and $\lambda\neq\lambda_0$, then $Q(\lambda,1)>\sqrt{f(\lambda)}\cdots(*)$, (ii) if $(*)$ is satisfied, then there is no real point on any small resolutions of Z_0.*

Proof. It is immediate to see that any real point of B is contained in some real plane $H_\lambda:=\{y_0=\lambda y_1\}$, $\lambda\in\mathbf{R}\cup\{\infty\}$. An equation of the restriction $B_\lambda:=B\cap H_\lambda$ is given by (as in the proof of Proposition 2.3) $\left(y_2y_3+Q(\lambda,1)y_1^2\right)^2-f(\lambda)y_1^4=0$. This can be rewritten as

$$B_\lambda:\left\{y_2y_3+\left(Q(\lambda,1)-\sqrt{f(\lambda)}\right)y_1^2\right\}\left\{y_2y_3+\left(Q(\lambda,1)+\sqrt{f(\lambda)}\right)y_1^2\right\}=0.$$

Namely, B_λ is a union of two conics. If $\lambda\neq\lambda_0$, we have $Q(\lambda,1)^2-f(\lambda)>0$ by our assumption, so both of $Q(\lambda,1)-\sqrt{f(\lambda)}$ and $Q(\lambda,1)+\sqrt{f(\lambda)}$ are non-zero.

Recall that our real structure is given by $\sigma(y_1:y_2:y_3)=(\overline{y}_1:\overline{y}_3:\overline{y}_2)$ on H_λ (Proposition 2.1, (ii)). Thus each component of B_λ is real iff the coefficients are real; namely $f(\lambda)\geq 0$. Further, the intersection of these two conics are $\{P_\infty,\overline{P}_\infty\}$. Therefore, there is no real point on B_λ if $f(\lambda)<0$. So suppose $f(\lambda)\geq 0$. In this case, each of the two conics are real, and by Lemma 2.4, both components have no real point iff $Q(\lambda,1)>\sqrt{f(\lambda)}$. On the other hand, we have $B_\infty=\{(y_2y_3+Q(y_0,0))^2=0\}$. Hence again by Lemma 2.4, we have $Q(1,0)>0$ if B_∞ has no real point. But this follows from the first condition. If $\lambda=\lambda_0$, we have $Q(\lambda_0,1)=\sqrt{f(\lambda_0)}$ and hence one of the components of B_{λ_0} degenerates into a union of two lines whose intersection is P_0. And the other component has no real point since $Q(\lambda_0,1)+\sqrt{f(\lambda_0)}=2\sqrt{f(\lambda_0)}>0$. Thus we get (i).

Next we show that Z_0 has no real point other than $p_0:=\Phi_0^{-1}(P_0)$, under the condition $(*)$. Suppose $y_1\neq 0$ and use the coordinate (v_0,v_2,v_3) defined in the proof of Proposition 2.2. Then Z_0 is given by the equation $z^2+(v_2v_3+Q(v_0,1))^2-f(v_0)=0$, where z is a fiber coordinate on the line bundle $O(2)$ over $\mathbf{CP}^3$. Thus to prove that the point p_0 is the only real locus of $Z_0\cap\Phi_0^{-1}(\{y_1\neq 0\})$, it suffices to show that

$$(v_2v_3+Q(v_0,1))^2-f(v_0)>0 \tag{5}$$

for any real $(v_0,v_2,v_3)\neq(\lambda_0,0,0)$. Recall that (v_0,v_2,v_3) is real iff $v_0\in\mathbf{R}$ and $v_2=\overline{v}_3$. Hence (5) is obvious for real (v_0,v_2,v_3) with $f(v_0)<0$. Assume $f(v_0)\geq 0$. Using the reality condition, we have $(v_2v_3+Q(v_0,1))^2-f(v_0)=|v_2|^4+2Q(v_0,1)|v_2|^2+(Q(v_0,1)^2-f(v_0))$. By our assumption we have $Q(v_0,1)^2-f(v_0)>0$ for any $v_0\in\mathbf{R}$ with $v_0\neq\lambda_0$. Further, by the condition $(*)$ we have $Q(v_0,1)>\sqrt{f(v_0)}$ for $v_0\neq\lambda_0$ with $f(v_0)\geq 0$. Therefore we have (5) also for real $(v_0,v_2,v_3)\neq(\lambda_0,0,0)$ with $f(v_0)\geq 0$. Thus we get that on $\Phi_0^{-1}(\{y_1\neq 0\})$, there are no real locus other than p_0. In the same way we can see that, over $y_0\neq 0$, there are no real points other than p_0. So it remains to show that there is no real point over

the line $l_\infty = \{y_0 = y_1 = 0\}$. To check this, we introduce a new homogeneous coordinate $(y_0, y_1, y_2 - y_3, y_2 + y_3)$ on $\mathbf{CP}^3$. Then the two subsets $y_2 - y_3 \neq 0$ and $y_2 + y_3 \neq 0$ are real, and it can be easily seen that over the line l_∞, the equation of Z_0 is of the form $z^2 + q^2 = 0$ on these two open subset, where q is non-zero real valued on the real set l_∞^σ. Hence Z_0 does not have real point over l_∞. Thus p_0 is the unique real point on the whole of Z_0.

Finally we show that $\Gamma_0 = \Phi^{-1}(P_0)$ has no real point. To show this, we use a coordinate (w_0, w_2, w_3) (around P_0) defined in (4). Then B is given by $w_1^2 + w_2 w_3 = 0$. Further, it is easy to see that the real structure is also given by $\sigma(w_1, w_2, w_3) = (\overline{w}_1, \overline{w}_3, \overline{w}_2)$. Now because Γ_0 is the exceptional curve of a small resolution of an ordinary double point, Γ_0 can be canonically identified with the set of lines contained in the cone $w_1^2 + w_2 w_3 = 0$. If $\{(w_1 : w_2 : w_3) = (a_1 : a_2 : a_3)\}$ is a real line, we can suppose $a_1 \in \mathbf{R}$, $a_3 = \overline{a}_2$. It follows that it cannot be contained in the cone. This implies that Γ_0 has no real point. On the other hand, resolutions of the singularities over P_∞ and $\overline{P}_\infty$ do not yield real points. Thus we can conclude that Z has no real point. □

Here we summarize necessary conditions for our threefolds to be (birational to) a twistor space:

Proposition 2.6. *Let B be a quartic surface in $\mathbf{CP}^3$ defined by*

$$(y_2 y_3 + Q(y_0, y_1))^2 - y_0 y_1 (y_0 + y_1)(y_0 - a y_1) = 0,$$

where $Q(y_0, y_1)$ is a quadratic form of y_0 and y_1 with real coefficients, and $a > 0$. Let $Z_0 \to \mathbf{CP}^3$ be the double covering branched along B. If there exists a resolution $Z \to Z_0$ such that Z is a twistor space, then Q and a satisfy the following condition:
Condition (A) *If λ satisfies $\lambda(\lambda + 1)(\lambda - a) \geq 0$, then*

$$Q(\lambda, 1) \geq \sqrt{\lambda(\lambda + 1)(\lambda - a)} \tag{6}$$

holds. Moreover, there exists a unique λ_0 such that the equality of (6) holds for $\lambda = \lambda_0$.

Proposition 2.1 (combined with some calculations given in §7.3) has the following consequence:

Proposition 2.7. *Let g be a self-dual metric on $3\mathbf{CP}^2$ of positive scalar curvature with a non-trivial Killing field, and assume that g is not conformally isometric to LeBrun metric. Then the naturally induced $U(1)$-action on $3\mathbf{CP}^2$ is uniquely determined up to diffeomorphisms.*

Proof. Let Z be the twistor space of g. Then Z is as in Proposition 2.1. Let H_i and T_i $(1 \leq i \leq 4)$ be as in the proof of Proposition 2.1. Namely, H_i is a real $U(1)$-invariant plane such that $B|_{H_i}$ is a trope whose reduction is denoted by T_i as before. Then $\Phi_0^{-1}(H_i)$ consists of two irreducible components, both of which are biholomorphic to H_i $(\simeq \mathbf{CP}^2)$. Since $\mu : Z \to Z_0$ is small, $\Phi^{-1}(H_i)$ also consists of two irreducible components, which are denoted by D_i and $\overline{D}_i$. These are $U(1)$-invariant. Then as will be proved in Lemma 7.9, there is a smooth rational surface D with $U(1)$-action such that D is $U(1)$-equivariantly biholomorphic to D_i for some $1 \leq i \leq 4$. Since $D_i + \overline{D}_i$ is a fundamental divisor, and since we have $-(1/2)K_Z \cdot L = 2$ (L is a twistor line), we have $D_i \cdot L = \overline{D}_i \cdot L = 1$. Hence by a result of Poon [15], $L_i := D_i \cap \overline{D}_i$ is a twistor line which is obviously $U(1)$-invariant, and that L_i is contracted to a point by the twistor fibration $Z \to 3\mathbf{CP}^2$ which is $U(1)$-equivariant. Hence the $U(1)$-action on $3\mathbf{CP}^2$ can be read from that on any one of D_i's. Therefore the conclusion of the proposition follows. □

The proposition implies that, up to diffeomorphisms, there are only two effective $U(1)$-actions on $3\mathbf{CP}^2$ which can be the identity component of the isometry group of a self-dual metric whose scalar curvature is positive. One is the semi-free $U(1)$-action, which is

the identity component of generic LeBrun metric, and the other is the action obtained in Proposition 2.7. Of course, there are many other differentiable $U(1)$-actions on $3\mathbf{CP}^2$ in general: for example, we can get an infinite number of mutually inequivariant $U(1)$-actions by first taking an effective $U(1) \times U(1)$-action on $3\mathbf{CP}^2$ and then choosing many different $U(1)$-subgroup of $U(1) \times U(1)$.

In the rest of this section we prove propositions which will be used in the subsequent sections.

Proposition 2.8. *Let B be the quartic surface defined by the equation (3) and suppose that Q and f satisfy the assumption in Proposition 2.5. Let $\Phi_0 : Z_0 \to \mathbf{CP}^3$ be the double covering branched along B, and $\mu : Z \to Z_0$ any small resolution preserving the real structure. Put $\Phi = \mu \cdot \Phi_0$. Then we have (i) $K_Z \simeq \Phi^* O(-2)$, (ii) the line bundle $(-1/2)K_Z$ is uniquely determined, (iii) Φ is the induced morphism associated to the complete linear system $|(-1/2)K_Z|$.*

Proof. Let K_{Z_0} denote the canonical sheaf of Z_0. Then we have $K_{Z_0} \simeq \Phi_0^*(K_{\mathbf{CP}^3} + (1/2)O(B)) \simeq \Phi_0^* O(-2)$. Moreover, since μ is small, we have $K_Z \simeq \mu^* K_{Z_0}$. Hence we get $K_Z \simeq \Phi^* O(-2)$. For (ii) it suffices to show that $H^1(O_Z) = 0$. Since the singularities of Z_0 are normal, and since the exceptional curves of μ are rational, we get by Leray spectral sequence $H^1(O_Z) \simeq H^1(O_{Z_0})$. Then applying the spectral sequence to Φ_0, and using $\Phi_{0*} O_{Z_0} \simeq O \oplus O(-2)$ and $R^q \Phi_{0*} O_{Z_0} = 0$ for $q \geq 1$, we get $H^1(O_{Z_0}) = 0$ and we get (ii). (iii) is easily obtained from (i). □

The following result will be also needed in the next section:

Proposition 2.9. *Let B be a quartic surface defined by (3). Then B does not contain real lines.*

Proof. Suppose that l is a real line lying on B. Because B is $\mathbf{C}^*$-invariant by the action $\rho_t : (y_0, y_1, y_2, y_3) \mapsto (y_0, y_1, ty_2, t^{-1}y_3)$ for $t \in \mathbf{C}^*$ (which is the complexification of the $U(1)$-action in Proposition 2.1 (iii)), $\rho_t(l)$ is a line contained in B. Hence if l is not $\mathbf{C}^*$-invariant, then it yields a one-parameter family of rational curves in B. This implies that B is a rational surface. However, B is birational to an elliptic ruled surface [16, 6]. Therefore l must be $\mathbf{C}^*$-invariant. It is readily verified from the above explicit form of the $\mathbf{C}^*$-action that $\mathbf{C}^*$-invariant real lines on $\mathbf{CP}^3$ are $l_0 = \{y_2 = y_3 = 0\}$ and $l_\infty = \{y_0 = y_1 = 0\}$ only. Both of these lines are clearly not contained in B. Hence there is no real line contained in B. □

The following 'octagon' will play a significant role throughout our investigation:

Proposition 2.10. *Let B, Z_0, Φ_0, μ and Φ be as in Proposition 2.8. Let l_∞ be a real line defined by $y_0 = y_1 = 0$. Then $\Phi^{-1}(l_\infty)$ is a cycle of eight smooth rational curves as in Figure 2, where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\overline{\Gamma} = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$ are the exceptional curves of the conjugate pair of singularities of Z_0, and Ξ and $\overline{\Xi}$ are conjugate pair of curves which are mapped biholomorphically onto l_∞.*

Proof. Let P_∞ and $\overline{P}_\infty$ be the conjugate pair of elliptic singularities of B as before and put $p_\infty = \Phi_0^{-1}(P_\infty)$ and $\overline{p}_\infty = \Phi_0^{-1}(\overline{P}_\infty)$ be the corresponding points of Z_0. Then since p_∞ and $\overline{p}_\infty$ are compound A_3-singularities of Z_0, the exceptional curve of a small resolution is a chain of three smooth rational curves. (See Section 7.3, where small resolutions are explicitly given. Another way to see this is that, cutting B by a generic plane containing l_∞ and taking the inverse image by Φ_0, one obtains a surface which has p_∞ and $\overline{p}_\infty$ as A_3-singularities. Then any small resolution of p_∞ and $\overline{p}_\infty$ necessarily gives the minimal resolution of the surface, by a well-known property of rational double points; cf. Proof of Lemma 4.6.) It remains to see how about the remaining components. It is readily seen that

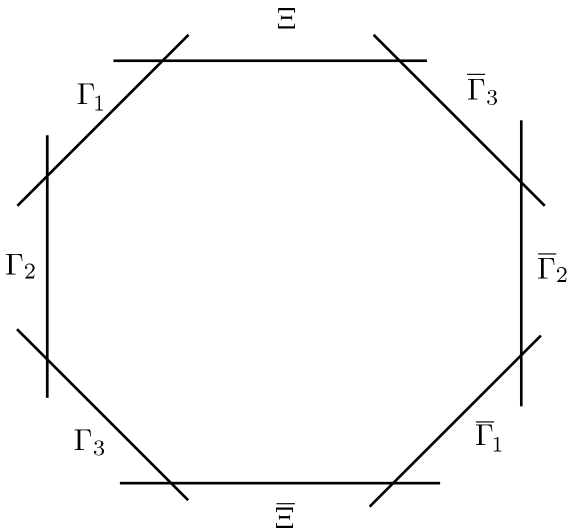

FIGURE 2. $\Phi^{-1}(l_\infty)$ becomes a cycle of 8 smooth rational curves

$\Phi_0^{-1}(l_\infty)$ consists of two smooth rational curves both of which are mapped biholomorphically onto l_∞. Moreover, as seen in the proof of Proposition 2.5, there is no real point on $\Phi_0^{-1}(l_\infty)$. Therefore the two components of $\Phi_0^{-1}(l_\infty)$ are conjugate of each other. Strictly speaking, we have to verify that the strict transforms in Z of these two components intersect Γ_1 and Γ_3 (and $\overline{\Gamma}_1$ and $\overline{\Gamma}_3$) as in Figure 2, and do not intersect Γ_2 and $\overline{\Gamma}_2$. But this can be checked by explicitly giving small resolutions as in Section 7.3 and chasing the inverse images. (Indeed this is essentially done in the proof of Lemma 7.19.) Thus we get the claim of the proposition. $\square$

3. How to find twistor lines

In the previous section we obtained necessary conditions (Proposition 2.6) of a quartic surface B to be the branch divisor of a twistor space of non-LeBrun self-dual metric on $3\mathbf{CP}^2$ with a non-trivial Killing field. The goal of this paper is to show that these are also sufficient conditions. We solve this problem by finding the family of twistor lines. In this section we will describe how to find twistor lines. The basic role is played by a classical result in algebraic geometry that a smooth plane quartic always has twenty-eight bitangents.

Let B be a quartic surface defined by the equation

$$(y_2y_3 + Q(y_0, y_1))^2 - y_0y_1(y_0 + y_1)(y_0 - ay_1) = 0 \tag{7}$$

and suppose that Conditions (A) for Q and a of Proposition 2.6 is satisfied. Let $P_0 = (\lambda_0, 1, 0, 0)$, $P_\infty = (0, 0, 0, 1)$ and $\overline{P}_\infty = (0, 0, 1, 0)$ be the singular locus of B, where P_0 is a real ordinary double point, and P_∞ and $\overline{P}_\infty$ are simple elliptic singularities of type $\tilde{E}_7$ (Proposition 2.2). Let $\Phi_0 : Z_0 \to \mathbf{CP}^3$ be the double covering branched along B. Let $\mu : Z \to Z_0$ be any one of small resolution of the three singular points of B, preserving the real structure of Z_0. (There are many choices of such μ. Later (in Sections 7 and 8) we will determine which resolution must be taken. Here we only suppose that μ preserves the real structure.) We do not assume that Z is a twistor space. Further we put $\Phi = \Phi_0 \cdot \mu$, and $\Gamma_0 = \Phi^{-1}(P_0), \Gamma = \Phi^{-1}(P_\infty)$ and $\overline{\Gamma} = \Phi^{-1}(\overline{P}_\infty)$ which are the exceptional curves of the small resolution of Z_0.

Set $H_1 = \{y_0 + y_1 = 0\}, H_2 = \{y_1 = 0\}, H_3 = \{y_0 - ay_1 = 0\}$ and $H_4 = \{y_0 = 0\}$. These are planes such that $H_i \cap B$ is a double conic (= trope), which is denoted by $T_i \,(\subset H_i)$.

Every H_i and T_i are real, where the real structure on $\mathbf{CP}^3$ is defined by

$$\sigma(y_0 : y_1 : y_2 : y_3) = (\overline{y}_0 : \overline{y}_1 : \overline{y}_3 : \overline{y}_2) \tag{8}$$

(cf. Proposition 2.1). Further, we write $\Phi^{-1}(H_i) = D_i + \overline{D}_i$ for $1 \le i \le 4$.

Definition 3.1. An irreducible conic C in $\mathbf{CP}^3$ is called a *touching conic* of B if $C \subset B$ or otherwise if the intersection number with B is even at any intersection points.

The following proposition motivates the definition:

Proposition 3.2. *Let L be a real line in Z. Then $\Phi(L)$ is a line in $\mathbf{CP}^3$ iff $L \cap \Gamma_0 \neq \phi$. Otherwise $\Phi(L)$ is a real touching conic of B.*

Note that any irreducible conic in $\mathbf{CP}^3$ is contained in a unique plane, and that the plane is real if the conic is real. In particular, it follows from Proposition 3.2 that any real line in Z always lies on the inverse image of some real plane.

Proof of Proposition 3.2. By adjunction formula, we have $-2 = K_Z \cdot L + \deg N_{L/Z} = K_Z \cdot L + 2$. Hence by Proposition 2.8 we have $(-1/2)K_Z \cdot L = 2$. Therefore $\Phi(L)$ is a curve whose degree is at most two, and $\Phi(L)$ is a line iff $\Phi|_L : L \to \Phi(L)$ is two-to-one.

Assume that $\Phi(L)$ is a line, which is necessarily real. By Proposition 2.9, $\Phi(L)$ is not contained in B. To prove $P_0 \in \Phi(L)$, it suffices to show that $\Phi(L) \cap B$ consists of three or one point, since by Proposition 2.5, P_0 is the unique real point of B. Since B is a quartic, $\Phi(L) \cap B$ consists of at most four points. If $\Phi(L) \cap B$ consists of four points, the intersection points are smooth points of B and $\Phi^{-1}(\Phi(L))$ must be a smooth elliptic curve, contradicting our assumption. If $\Phi(L) \cap B$ consists of two points, the intersection points must be singular points of B, since otherwise $\Phi^{-1}(\Phi(L))$ would split into two rational curves, which contradicts that $\Phi|_L : L \to \Phi(L)$ is two to one. Therefore the two intersection points must be singular points of B and it follows $\Phi(L) = l_\infty$. However, by Proposition 2.10, $\Phi^{-1}(l_\infty)$ does not contain real components. (Or more strongly any component of $\Phi^{-1}(\Phi(L))$ does not mapped two-to-one onto its image.) Hence $\Phi(L) \cap B$ cannot consist of two points. Therefore it must consists of three or one point (although the latter cannot happen as is readily seen), and we have $P_0 \in \Phi(L)$. It follows that $L \cap \Gamma_0 \neq \phi$. Conversely assume that L is a real line intersecting Γ_0. Then since there are no real points on Γ_0 (Proposition 2.5 (ii)), the intersection is not one point. Because $\Phi(\Gamma_0) = P_0$, this implies that Φ is not one-to-one on L. Hence $\Phi(L)$ must be a line.

Finally suppose that $\Phi(L)$ is a conic. If $(\Phi(L), B)_P = 1$ for some $P \in \Phi(L) \cap B$, then P is a smooth point of B and the intersection is transversal. Therefore $\Phi^{-1}(\Phi(L))$ is locally irreducible near $\Phi^{-1}(P)$. This contradicts to the fact that $\Phi|_L$ is bijective. Therefore we have $(\Phi(L), B)_P \ge 2$ for any $P \in \Phi(L) \cap B$. Moreover, $(\Phi(L), B)_P$ must be even, since otherwise $\Phi^{-1}(\Phi(L))$ would be locally irreducible. This implies that $\Phi(L)$ is a touching conic of B, which is necessarily real. $\square$

In the sequel $(\mathbf{CP}^3)^\vee$ denotes the dual complex projective space, which can be viewed as the set of planes in $\mathbf{CP}^3$. Let $(\mathbf{RP}^3)^\vee \subset (\mathbf{CP}^3)^\vee$ be the dual projective space of $\mathbf{RP}^3$, which is the set of real planes in $\mathbf{CP}^3$, where we are assuming that the real structure on $\mathbf{CP}^3$ is given by (8). According to Proposition 2.8, for $H \in (\mathbf{RP}^3)^\vee$, $S_H := \Phi^{-1}(H)$ is a real member of $|(-1/2)K_Z|$. If Z is actually a twistor space, the following basic result of Pedersen-Poon [12] concerning the structure of real members of this complete system is known:

Proposition 3.3. *Any real irreducible member S of $|(-1/2)K_Z|$ of a compact twistor space is a smooth surface. Moreover, such an S always contains a real pencil whose real members (which are automatically parametrized by a circle) are just the set of twistor lines of Z contained in S. Further, each member has the trivial normal bundle in S.*

This provides a necessary condition for a threefold to be a twistor space. However, for a general twistor space, it is not true that generic twistor lines are contained in some $S \in |(-1/2)K|^\sigma$. An important point in the present case is that this is true, due to the remark after Proposition 3.2. We will see in Proposition 3.5 that $S_H = \Phi^{-1}(H)$ is smooth for any $H \in (\mathbf{RP}^3)^\vee$, unless $\Phi^{-1}(H)$ is reducible (i.e. $H \neq H_i$, $1 \le i \le 4$).

By using the $\mathbf{C}^*$-action, we can determine real planes H for which $B \cap H$ is singular:

Lemma 3.4. *For a real plane $H \in (\mathbf{RP}^3)^\vee$, $B \cap H$ is a singular quartic precisely when H goes through a singular point of B. Moreover, any singular point of $B \cap H$ must be a singular point of B.*

Proof. Recall that B is invariant by the $\mathbf{C}^*$-action $(y_0, y_1, y_2, y_3) \mapsto (y_0, y_1, ty_2, t^{-1}y_3)$, $t \in \mathbf{C}^*$, and that B has only isolated singular points, which are P_0, P_∞ and $\overline{P}_\infty$. It follows that any singular point of $H \cap B$ is a $\mathbf{C}^*$-fixed point, since otherwise B would have singularities along the $\mathbf{C}^*$-orbit. It is immediate to see that the $\mathbf{C}^*$-fixed set on $\mathbf{CP}^3$ is a line $l_0 = \{y_2 = y_3 = 0\}$ and two points P_∞ and $\overline{P}_\infty$. But it is readily seen that $B \cap l_0$ consists of three points, one of which is P_0 and the other two are a pair of conjugate points (cf. Proof of Proposition 8.1). It follows from the reality of H that if H contains one of the conjugate pair of points, then H contains l_0 and in particular contains P_0. Thus we have seen that $H \cap B$ is singular precisely when H goes through P_0, P_∞ or $\overline{P}_\infty$ and we obtain the former claim of the lemma. Moreover, the conjugate pair of points of $B \cap l_0$ cannot be singular points of $B \cap H$ ($H \supset l_0$), since B is a quartic and P_0 is a double point. Thus any singular point of $H \cap B$ must be P_0, P_∞ or $\overline{P}_\infty$ and we get the latter claim of the lemma. $\square$

Next we introduce some notations that will be frequently used in the sequel. We set $U := \{H \in (\mathbf{CP}^3)^\vee \,|\, H \cap B \text{ is a non-singular quartic}\}$, which is a Zariski-open subset of $(\mathbf{CP}^3)^\vee$. Set $U^\sigma = U \cap (\mathbf{RP}^3)^\vee$. By Proposition 3.4, U^σ is the set of planes not going through P_0, P_∞ and $\overline{P}_\infty$. From this, it is immediate to see that $(\mathbf{RP}^3)^\vee \backslash U^\sigma$ consists of two components: one is the set of real planes going through P_0 and another is the set of real planes containing $l_\infty = \{y_0 = y_1 = 0\}$. We denote by $\mathbf{RP}^2_\infty$ for the former component and $\langle l_\infty \rangle^\sigma$ for the latter component respectively. Note that $\mathbf{RP}^2_\infty \cap \langle l_\infty \rangle^\sigma$ is a single plane, which we will denote by H_{λ_0} (since it is defined by $y_0 = \lambda_0 y_1$, where λ_0 appears in Condition (A) of Proposition 2.6) . Members of $\langle l_\infty \rangle^\sigma$ can be characterized by the $\mathbf{C}^*$-invariance, where the $\mathbf{C}^*$-action is as in the proof of Lemma 3.4. $\mathbf{RP}^2_\infty$ can be considered as a plane at infinity on $(\mathbf{RP}^3)^\vee$, and by removing it, we get an Euclidean space $\mathbf{R}^3$. Then $\langle l_\infty \rangle^\sigma$ is a line in this $\mathbf{R}^3$, and U^σ is isomorphic to $\mathbf{R}^3 \backslash \mathbf{R} \simeq \mathbf{C}^* \times \mathbf{R}$. In particular we have $\pi_1(U^\sigma) \simeq \mathbf{Z}$. This fact will play an essential role in our global construction of twistor lines given in Section 9.

The following proposition implies that our threefold Z actually satisfies a necessary condition to be a twistor space imposed by Proposition 3.3.

Proposition 3.5. *If H is a real plane different from H_i ($1 \le i \le 4$), then $S_H = \Phi^{-1}(H)$ is a smooth rational surface satisfying $c_1^2(S_H) = 2$.*

Proof. Recall that $\Phi_0 : Z_0 \to \mathbf{CP}^3$ is the double covering branched along B and $\Phi : Z \to \mathbf{CP}^3$ is the composition of Φ_0 with a small resolution of Z_0. Thus if H is a real plane not going the singular points of B, $S_H \to H$ is simply a double covering whose branch is $B \cap H$. By Lemma 3.4, this kind of H is exactly real planes such that $B \cap H$ is a smooth quartic; namely $H \in U^\sigma$. Hence $S_H \to H$ is a double covering branched along a smooth quartic. It follows that S_H is a smooth rational surface with $c_1^2 = 2$ (which is a del-Pezzo surface). Thus the claim is proved for $H \in U^\sigma$. Next suppose that $H \in \mathbf{RP}^2_\infty$ and $H \neq H_{\lambda_0}$. Then $H \cap B$ is a quartic curve whose singular points is P_0 only by Lemma 3.4. $\Phi_0^{-1}(H)$ has a unique ordinary point over P_0. But this is always resolved through any small resolution $Z \to Z_0$. Hence S_H is again smooth. Moreover, S_H must be a rational surface with $c_1^2 = 2$, since by moving the plane H to be a member of U^σ, S_H can be considered as a deformation of $S_{H'}$ for

$H' \in U^\sigma$ and since $S_{H'}$ is a rational surface with $c_1^2 = 2$ as is already mentioned. (However, this time S_H is not a del-Pezzo surface any more.) Thus the claim of the proposition is also proved for $H \in \mathbf{RP}^2_\infty$. Finally suppose that $H \in \langle l_\infty \rangle^\sigma$. Since H is $\mathbf{C}^*$-invariant, $H \cap B$ must be also $\mathbf{C}^*$-invariant. It is readily seen that $H \cap B$ is a union of two $\mathbf{C}^*$-invariant conics going through P_∞ and $\overline{P}_\infty$, and the two conics are mutually different iff $H \neq H_i$ ($1 \leq i \leq 4$). It follows that $\Phi_0^{-1}(H)$ has A_3-singularities over P_∞ and $\overline{P}_\infty$. These singularities are also resolved by any small resolution of the conjugate pair of singular points of Z_0, and therefore S_H is smooth. (These are proved Lemma 7.11 by direct calculation using local coordinates.) The structure of S_H is again as in the claim of the proposition by the same reason for the previous case, and we have obtained all of the claims. □

Our next task is to find twistor lines lying on S_H. Again by Proposition 3.3 twistor lines on S_H must be the real part of a real pencil on S_H whose corresponding line bundle $\mathcal{L}_H \to S_H$ satisfies $(\mathcal{L}_H)^2_S = 0$. There are many real line bundles $\mathcal{L}$ on S satisfying $(\mathcal{L})^2_S = 0$. However, we know by Proposition 3.2 that the image of general lines is a touching conic. Thus we need to know about touching conics lying on real planes.

First we consider the case that the quartic curve is smooth, and forget the reality for a while (until the proof of Proposition 3.10). We recall the following classical result in algebraic geometry, which follows from the Plücker formula.

Proposition 3.6. *Any smooth plane quartic has 28 bitangents.*

Next we see how the bitangents generate touching conics. For this purpose, let B_1 be a smooth plane quartic and $\phi : S \to \mathbf{CP}^2$ the double covering branched along B_1. It is easily seen that $-K_S \simeq \phi^* O(1)$ and that S is a del-Pezzo surface of degree two. The following result is also well-known and easy to prove:

Proposition 3.7. *(i) The inverse image of any bitangent of a smooth plane quartic consists of two (-1)-curves E_1 and E_2 on S, both of which are mapped biholomorphically onto the original bitangent, (ii) $E_1 \cap E_2$ consists of two points and the intersections are transversal, (iii) the image of any (-1)-curve on S is a bitangent.*

It follows from Propositions 3.6 and 3.7 that the number of (-1)-curves on S is $28 \times 2 = 56$.

Lemma 3.8. *Let B_1 and $\phi : S \to \mathbf{CP}^2$ be as above and C a touching conic of B_1. Then (i) $\phi^{-1}(C)$ is a reducible curve whose irreducible components are two smooth rational curves which are mapped biholomorphically onto C by ϕ, (ii) the two irreducible components of $\phi^{-1}(C)$ have trivial normal bundles in S, and they define two pencils on S, (iii) the images of general members of the two pencils in (ii) are touching conics of B_1, and they define a one-dimensional family of touching conics.*

Proof. Let $P \in B_1 \cap C$. Then by definition of touching conic (Definition 3.1), we can suppose that in a neighborhood of P, B_1 and C are defined by $x^{2m} - y = 0$ and $y = 0$ respectively for some $m \geq 1$. Then on the inverse image S is locally defined by $z^2 = x^{2m} - y$ and hence $\phi^{-1}(C)$ is locally defined by $z^2 = x^{2m}$. It follows that $\Phi^{-1}(C)$ is locally reducible at $\phi^{-1}(P)$. Let $(\phi^{-1}(C))' \to \phi^{-1}(C)$ be the normalization. Then the naturally induced map $(\phi^{-1}(C))' \to C$ is an unramified double covering of C. Therefore $(\phi^{-1}(C))'$ is a disjoint union of two rational curves, since C is simply connected. This implies that $\phi^{-1}(C)$ is reducible. We write $\phi^{-1}(C) = L_1 + L_2$. It is obvious L_1 and L_2 are smooth rational curves which are mapped biholomorphically onto C. Thus we have (i). Since $L_1 + L_2 \in |\phi^* O(2)|$ and since $(\phi^* O(1))^2 = c_1^2(S) = 2$, we have $(L_1 + L_2)^2 = 8$ on S. Moreover, by the above local description of $\phi^{-1}(C)$ near $P \in B_1 \cap C$, and recalling that $(C, B_1) = 2 \cdot 4 = 8$ on $\mathbf{CP}^2$, it follows that $(L_1, L_2) = 4$ on S. Therefore we have $L_1^2 + L_2^2 = 0$ on S. Furthermore, because S is a del-Pezzo surface, there is no smooth rational curve whose self-intersection number on S is less than -2. Further, since the image of (-1)-curves are bitangents (Proposition

3.7), L_1 and L_2 are not (-1)-curves. Thus we have deduced $L_1^2 = L_2^2 = 0$. It is obvious that $|L_1|$ and $|L_2|$ are pencils on S, and $|L_1| \neq |L_2|$ since $L_1L_2 = 4$. Thus we get (ii). Finally L_1 and L_2 respectively define pencils whose self-intersections are zero, and L_1 and L_2 are a smooth member of the pencil respectively. By taking the image of the members of the pencils, we get a one-dimensional family of conics parametrized by $\mathbf{CP}^1$. It remains to see that these are touching conics. Let $L \subset S$ be a generic member of the pencil $|L_1|$. Then since the property that $\phi|_L : L \to \phi(L)$ is isomorphic is an open condition, $\phi|_L$ is isomorphic onto the image. Therefore as in the final step in the proof of Proposition 3.2, $\phi(L)$ must be a touching conic. □

Next we investigate the number of (one-dimensional) families of touching conics appeared in Lemma 3.8. So let C be a touching conic and write $\phi^{-1}(C) = L_1 + L_2$ as in the proof.

Lemma 3.9. *(i) Both of the pencils $|L_1|$ and $|L_2|$ contain exactly 6 reducible members, any of which are the sum of two (-1)-curves intersecting transversally at a point. (ii) Each one-dimensional family of touching conics contains exactly 6 reducible members, which are the sum of two bitangents.*

Proof. (i) Because S is a del-Pezzo surface, there is no smooth rational curve whose self-intersection is less than -1. Therefore any reducible member of $|L_1|$ is of the form in the lemma. Since we have $c_1^2(S) = 2$, the number of such reducible members must be 6. (ii) is immediate from (i) and Proposition 3.7. □

Proposition 3.10. *A smooth plane quartic has precisely 63 (one-dimensional) families of touching conics.*

Proof. Let l and l' be any two different bitangents. Then by Proposition 3.7 we can write $\phi^{-1}(l) = E_1 + E_2$ and $\phi^{-1}(l') = E_1' + E_2'$, where E_1, E_2, E_1', E_2' are (-1) curves, and E_1 and E_2 (and E_1' and E_2' also) intersect transversally at two points. Because $l \cap l'$ is not on the branch quartic, we can suppose that $E_1E_1' = E_2E_2' = 1$ and $E_1E_2' = E_2E_1' = 0$ (Figure 3). Then we have $(E_1 + E_1')^2 = (E_2 + E_2')^2 = 0$. From this it can be easily shown that $|E_1 + E_1'|$ and $|E_2 + E_2'|$ are pencils on S whose general members are smooth rational curves. Taking the image by ϕ, we get a one-dimensional family of touching conics, which contains $l + l'$ as one of the 6 reducible members. In this way a choice of two bitangents defines a (one-dimensional) family of touching conics. By Proposition 3.6 there are $28!/2!26! = 378$ such choices. Moreover we know by Lemma 3.9 that each family contains just 6 reducible members. Therefore, the number of the families must be $378/6 = 63$. □

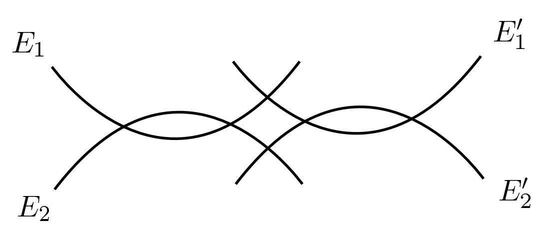

FIGURE 3. the inverse image of a pair of bitangents

So far we have considered bitangents and touching conics of a fixed smooth plane quartic. From now on we go back to the situation where B is a real quartic surface defined by (7) and consider the space of touching conics of B. Let $\mathcal{P} \to (\mathbf{CP}^3)^\vee$ be the $\mathbf{CP}^5$-bundle whose fiber over $H \in (\mathbf{CP}^3)^\vee$ is the space of conics on H. Let $\mathcal{P}_U \to U$ be the restriction onto U and $\mathcal{C} \subset \mathcal{P}_U$ the closed subset formed by touching conics. Since B has a real structure, $\mathcal{C}$ also has a real structure, and real members of $\mathcal{C}$ are candidates of the image of twistor lines. Then the following proposition is obvious from Proposition 3.10:

Proposition 3.11. *The natural projection $\mathcal{C} \to U$ is a fiber bundle whose fibers are disjoint union of 63 copies of* $\mathbf{CP}^1$.

Now it is ready to explain how we will obtain all the twistor lines in Z. (A complete proof will be given in Sections 9 and 10.) By Lemma 3.8, $\mathcal{C}$ admits a natural double covering $\tilde{\mathcal{C}} \to \mathcal{C}$ whose fiber over $C \in \mathcal{C}$ represents the two irreducible components of $\Phi^{-1}(C)$. The composition $\tilde{\mathcal{C}} \to \mathcal{C} \to U$ is a fiber bundle whose fibers are 126 copies of $\mathbf{CP}^1$. Because Z_0 has a real structure, the real structure on $\mathcal{C}$ naturally lifts on $\tilde{\mathcal{C}}$. Let $\tilde{\mathcal{C}}_{U^\sigma} \to U^\sigma$ be the restriction of $\tilde{\mathcal{C}} \to U$ onto U^σ. Because $U^\sigma \simeq \mathbf{C}^* \times \mathbf{R}$, a monodromy problem naturally arises for this bundle, and generic twistor lines in Z must be real members of $\tilde{\mathcal{C}}_{U^\sigma}$ which are monodromy invariant. Hence studying monodromy of $\tilde{\mathcal{C}}_{U^\sigma} \to U^\sigma$ is one way to detect generic twistor lines in Z. However, we do not take this direction and instead investigate what happens as a plane $H \in U^\sigma$ moves to be in $\langle l_\infty \rangle^\sigma$. For any $H \in \langle l_\infty \rangle^\sigma$, we will determine real lines lying on $S_H = \Phi^{-1}(H)$ in explicit form. This is accomplished at length in Section 5–7. Then we will show that there exists a neighborhood W of $\langle l_\infty \rangle^\sigma$ in $(\mathbf{RP}^3)^\vee$ such that for any $H' \in W$, S^1-family of real lines on $S_{H'}$ is obtained as a unique extension of the S^1-family of real lines on S_H, $H \in \langle l_\infty \rangle^\sigma$. Real lines obtained in this ways become automatically invariant by the monodromy action, since a loop around $\langle l_\infty \rangle^\sigma$ in W generates $\pi_1(U^\sigma) \simeq \mathbf{Z}$. Moreover, these real lines extend in a unique way to give a S^1-family of real smooth rational curves on S_H for any $H \in U^\sigma$. On the other hand, we will determine in Section 8, S^1-family of real lines lying on S_H for $H \in \mathbf{RP}^2_\infty$. Further, we will show that these real lines on S_H, $H \in \mathbf{RP}^2_\infty$ is a deformation of real rational curves obtained for $H \in U^\sigma$. In this way we will get a connected four-dimensional family of real rational curves in Z.

Once candidates of twistor lines are determined, we next have to show that these curves foliate Z. In Section 9, we show that this is the case; namely we prove that, for any point of Z, there uniquely exists a member of the family going through the point. We have to be careful at this point since as will be proved in Section 7.2, there exists a connected family of real lines in Z such that different members of the family really intersect (Proposition 7.8). In our proof of the disjointness of lines, we will find the following interesting geometric obstruction for B to define a twistor space. In order to explain that, recall that H_i ($1 \le i \le 4$) are the real planes on which the tropes T_i lie. Then any line on H_i is automatically a bitangent.

Definition 3.12. We call lines on H_i *trivial bitangents*. Bitangent of B which is not trivial is called a *non-trivial bitangent*.

Evidently, the space of trivial bitangents consists of four irreducible components, which are $H_i^\vee$ (the dual projective plane). By configuration of $\{H_i\}$ we have $(H_i)^\vee \cap (H_j)^\vee = \{l_\infty\}$ for any $i \neq j$. On any plane other than H_i there are just four trivial bitangents which are the intersection of the plane with H_i. If the plane is real, these trivial bitangents are obviously real. The following proposition implies that non-trivial real bitangent becomes an obstruction to define a twistor space:

Proposition 3.13. *If B has a non-trivial real bitangent, then there is no resolution $Z \to Z_0$, such that Z is a twistor space.*

Proof. Let l be a non-trivial real bitangent. Since l_∞ is a trivial bitangent, $l \neq l_\infty$. Further, a real line going through P_0 cannot be a bitangent since by Proposition 2.5, P_0 is the unique real point of B and hence the other touching point cannot be real. Thus l does not go through the singular points P_0, P_∞ and $\overline{P}_\infty$. Let P and $\overline{P}$ be the touching points of l with B, and p and $\overline{p}$ the corresponding points of Z respectively. Suppose that Z is a twistor space and take the twistor line $L \subset Z$ joining p and $\overline{p}$. Then $\Phi(L)$ must be a conic, since $\Phi^{-1}(l)$ cannot contain a twistor line. Hence by Proposition 3.2, $\Phi(L)$ must be a real conic

and there exists a unique real plane H containing $\Phi(L)$. This H does not go through P_0 since the image of a twistor line lying on the inverse image of such a real plane must be a line by Proposition 3.2. Assume that H goes through P_∞. Then H contains l_∞ by reality. Namely, $H \in \langle l_\infty \rangle^\sigma$. Hence $H \cap B$ is a union of two irreducible conics which are the closure of $\mathbf{C}^*$-orbits. Moreover, since l is not a trivial bitangent, $H \neq H_i$ for any $1 \leq i \leq 4$. Hence the two conics are mutually different. However, as will be seen in Proposition 4.5, a quartic of this kind (a union of these two conics) does not have real bitangent other than l_∞. This contradicts our choice of l. Hence we can suppose that H does not go through P_∞ (and $\overline{P}_\infty$). Then by Lemma 3.4, $H \cap B$ is smooth and S_H (the double cover of H branched along $H \cap B$) can be considered as a smooth surface in Z. By our construction S_H contains both of L and $\Phi^{-1}(l)$, and the latter is an anticanonical curve of S_H which is a sum of two (-1)-curves intersecting transversally at p and $\overline{p}$. Thus the intersection number of L and $\Phi^{-1}(L)$ on S_H must be at least 4 (since L goes through P and $\overline{P}$). On the other hand since $L^2 = 0$ on S_H (see Proposition 3.3), we have $-2 = K_S L + L^2 = K_S L$ and hence $-K_S L = 2$. This is a contradiction, and hence there cannot exist non-trivial real bitangents, if Z is a twistor space. □

But fortunately, we have the following

Proposition 3.14. *There is no non-trivial real bitangent of B.*

In the next section we prove this proposition by studying degeneration of quartic curves and their bitangents.

4. Behavior of bitangents as a plane quartic degenerates

In this section we investigate the behavior of bitangents of plane quartics when the planes move to contain the line l_∞. As a result we prove Proposition 3.14 saying that our branch quartic surface B does not have non-trivial bitangents (see Definition 3.12). As seen in Proposition 3.13, non-trivial real bitangents are obstruction for the threefold Z to be a twistor space and non-existence is an important step in our proof of the main theorem.

Let us first consider the simplest situation when a plane quartic degenerates into a quartic which has a node as its only singularity. Although we do not need this case itself, clarifying the behavior of bitangents in this simplest case will help to understand the case we really want.

As in Proposition 3.6, a smooth plane quartic has 28 bitangents. Plücker formula also implies that a quartic with a unique node has 16 bitangents. However, the formula only counts the number of bitangents which do not go through the node; namely it does not count lines going through the node and touching the quartic at a different point. We can count the number of such bitangents:

Proposition 4.1. *A plane quartic B_0 having a node as its only singularity possesses precisely 6 bitangents going through the node.*

Proof. Let P_0 be the unique node of B_0. Let $\phi_0 : S_0 \to \mathbf{CP}^2$ be the double covering whose branch is B_0. S_0 has an ordinary double point over P_0. Let $S_0' \to S_0$ be the minimal resolution and E the exceptional curve which is a smooth rational curve on S_0' whose self-intersection is -2. We denote by $\psi : S_0' \to \mathbf{CP}^2$ the composition of ϕ_0 with the resolution. S_0' is a smooth rational surface with $c_1^2 = 2$, and we have $-K \simeq \psi^* O(1)$. Hence we have $(\psi^* O(1) - E)^2 = c_1^2 + E^2 = 0$ on S_0'. If l is a line going through P_0 and intersecting B_0 at other two different points, $\psi^{-1}(l) - E$ is a smooth member of the system $|\psi^* O(1) - E|$ and it follows that $|\psi^* O(1) - E|$ is a pencil whose general members are smooth rational curves. Moreover, even if l coincides with one of the two tangent lines of B_0 at P_0, $\psi^{-1}(l) - E$ is still a smooth rational curve (in this case $\psi^{-1}(l) - E$ touches E), and these two tangent lines also give smooth members of $|\psi^* O(1) - E|$. Remaining is the case that l being a line

through P_0 but touches B_0 at a different point; namely l is a bitangent through P_0. If l is this kind of bitangent, $\phi_0^{-1}(l) - E$ consists of two smooth rational curves intersecting at two points, one of which is the ordinary double point of S_0 and the other is over the touching point. Thus $\psi^{-1}(l)$ is a triangle of smooth rational curves such that every three intersection is transversal (Figure 4). Write Ξ_1 and Ξ_2 for the two components other than E. $\Xi_1 + \Xi_2$ is a reducible member of $\psi^{-1}(l) - E$ and we have $\Xi_1^2 = \Xi_2^2 = -1$ on S_0'. Thus $|\psi^* O(1) - E|$ is a pencil whose reducible members consist of two irreducible components. Because we know $c_1^2(S_0') = 2$, it follows that the number of reducible members must be six. This implies that B_0 has exactly 6 bitangents going through the node. □

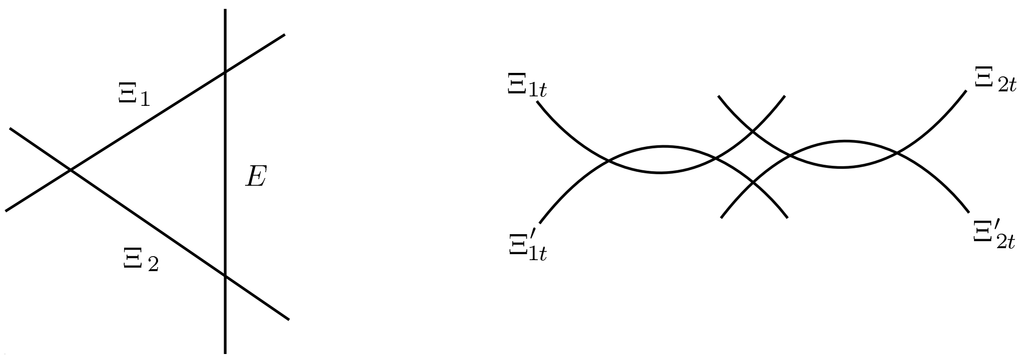

FIGURE 4. the inverse image of a bitangent through the node (left) and its small deformation (right)

Thus B_0 has $16 + 6 = 22$ bitangents in all. We next investigate the relationship of these bitangents with 28 bitangents of a smooth quartic. Let Δ be a unit disk in $\mathbf{C}$ and $\mathcal{B}_1 \subset \mathbf{CP}^2 \times \Delta$ a complex subspace such that $B_t := \mathcal{B}_1 \cap (\mathbf{CP}^2 \times \{t\})$ is a smooth quartic for any $t \neq 0$ and such that $B_0 = \mathcal{B}_1 \cap (\mathbf{CP}^2 \times \{0\})$ is a quartic which has a node as its only singularity. Then $\mathcal{B}_1$ has a unique singular point which is the node of B_0. By utilizing a projection $\mathcal{B}_1 \to \Delta$ we consider $\mathcal{B}_1$ as a one-dimensional family of plane quartics. Then 16 bitangents of B_0 not going through P_0 clearly extend in a unique way to give 16 bitangents of B_t for $t \neq 0$. The following lemma describes how the remaining 12 $(= 28 - 16)$ bitangents of B_t are obtained from the 6 bitangents of B_0 going through the node.

Lemma 4.2. *Among 28 bitangents of a smooth plane quartic B_t $(t \neq 0)$, there are 12 bitangents with the following property: the 12 bitangents form 6 pairs and the two members of the same pair will converge, as $t \to 0$, to the same bitangents of B_0 going through the node.*

Of course, the 12 bitangents in the proposition depend on a choice of degeneration $\mathcal{B}_1 \to \Delta$.

Proof of Lemma 4.2. Let l be a bitangent of B_0 going through the node. Let $\mathcal{S}_1 \to \mathbf{CP}^2 \times \Delta$ be the double covering branched along $\mathcal{B}_1$ and $\mathcal{S}_1'$ a small resolution of the ordinary double point of $\mathcal{S}_1$. By composition $\mathcal{S}_1' \to \mathcal{S}_1 \to \mathbf{CP}^2 \times \Delta \to \Delta$, we get a projection $\mathcal{S}_1' \to \Delta$ whose fibers are smooth rational surfaces with $c_1^2 = 2$. This can be viewed as a degeneration of del-Pezzo surface of degree two. Let $\psi : S_0' \to \mathbf{CP}^2$ denote the restriction of $\mathcal{S}_1' \to \mathbf{CP}_2 \times \Delta$ onto the central fiber $(=\mathbf{CP}^2 \times \{0\})$. Then we can write $\psi^{-1}(l) = E + \Xi_1 + \Xi_2$ which is a triangle of smooth rational curve, where E is the exceptional (-2)-curve, and Ξ_1 and Ξ_2 are (-1)-curves as in the proof of Proposition 4.1. Then since (-1)-curves survive under small deformation of surface, Ξ_1 and Ξ_2 extend to be (-1)-curves on S_t (= the fiber of $\mathcal{S}_1' \to \Delta$ over t) for sufficiently small $t \in \Delta$. Moreover, such extensions are obviously unique. Let Ξ_{1t} and Ξ_{2t} be the (-1)-curves on S_t obtained in this way. The stability of (-1)-curves

only guarantees the existence of such Ξ_{1t} and Ξ_{2t} for small t, but these extend to any $t \in \Delta$ since these are irreducible components of the inverse image of bitangents. Since $\Xi_1\Xi_2 = 1$ on S'_0, we have $\Xi_{1t}\Xi_{2t} = 1$ on S_t. Taking the image onto $\mathbf{CP}_2$, we get a pair of bitangents of B_t, $t \neq 0$. These bitangents are mutually different since if they coincide, the inverse image must be a pair of (-1)-curves intersecting transversally at two points over the touching points (Proposition 3.7), which contradicts $\Xi_{1t}\Xi_{2t} = 1$. Thus we have seen that a bitangent through the node generates two bitangents of a smooth quartic. Moreover it is obvious that different bitangents cannot generate the same bitangent. Hence we get $2 \times 6 = 12$ bitangents of a smooth quartic. $\square$

As in the above proof of Lemma 4.2, we have two (-1)-curves Ξ_{1t} and Ξ_{2t} of S_t which converge to Ξ_1 and Ξ_2 as $t \to 0$ respectively. One may wonder how about the limits of the other components $\phi_t^{-1}(\phi_t(\Xi_{1t})) - \Xi_{1t}$ and $\phi_t^{-1}(\phi_t(\Xi_{2t})) - \Xi_{2t}$, where $\phi_t : S_t \to \mathbf{CP}_2$ is the natural projection. At this point, we have the following

Lemma 4.3. *When t goes to 0 in Δ, then $\phi_t^{-1}(\phi_t(\Xi_{1t})) - \Xi_{1t}$ and $\phi_t^{-1}(\phi_t(\Xi_{2t})) - \Xi_{2t}$ respectively converge to reducible curves $E + \Xi_2$ and $E + \Xi_1$ in S'_0.*

Proof. This is also proved by standard argument in deformation theory. In this proof we write $S = S'_0$ for simplicity. Let F be the line bundle on S associated to the divisor $E + \Xi_2$. We readily have $F^2 = -1$ on S. The exact sequence $0 \to O_S \to O_S(\Xi_2) \to O_{\Xi_2}(\Xi_2) \to 0$ and $\Xi_2^2 = -1$ and the rationality of S imply $H^i(O_S(\Xi_2)) = 0$ for $i = 1, 2$. Then by the exact sequence $0 \to O_S(\Xi_2) \to F \to O_E(\Xi_2 + E) \to 0$ and $E\Xi_2 = 1$ and $E^2 = -2$, we get $H^0(S, F)$ is one-dimensional and $H^i(S, F) = 0$ for $i = 1, 2$. Since S is a rational surface, the line bundle $F \to S$ extends in a unique way to give a line bundle $F_t \to S_t$ for $t \neq 0$. Then we have $H^i(S_t, F_t) = 0$ for $i = 1, 2$ by the upper-semicontinuity and $\dim H^0(S_t, F_t) = 1$ by the invariance of the Euler characteristic $\chi(F_t)$ under deformation. In particular $|F_t|$ still consists of a unique member. Obviously $F_t^2 = -1$ on S_t. Then because S_t is a del-Pezzo surface for $t \neq 0$, there does not exist a smooth rational curve on S_t whose self-intersection is less than -1. This implies that the unique member of $|F_t|$ ($t \neq 0$) must be irreducible and it must be a (-1)-curve on S_t. Thus we get a (-1)-curve Ξ'_{1t} on S_t as a deformation of $E + \Xi_2$. To complete a proof of the lemma it suffices to show that $\Xi'_{1t} = \phi_t^{-1}(\phi_t(\Xi_{1t})) - \Xi_{1t}$. It is immediate to verify that $\Xi_1(E + \Xi_2) = 2$ on S. Hence we have $\Xi_{1t}\Xi'_{1t} = 2$ on S_t. Namely $\{\Xi_{1t}, \Xi'_{1t}\}$ is a pair of (-1)-curves satisfying $\Xi_{1t}\Xi'_{1t} = 2$. On the other hand, in view of Proposition 3.7, the intersection number of different (-1)-curves on S_t is $0, 1$ or 2, and it becomes two exactly when their images are the same bitangent (see Figure 4). Thus we have $\phi_t(\Xi_{1t}) = \phi_t(\Xi'_{1t})$. Therefore we obtain $\Xi'_{1t} = \phi_t^{-1}(\phi_t(\Xi_{1t})) - \Xi_{1t}$, as claimed. The claim for $E + \Xi_1$ is completely parallel. $\square$

The following lemma is obvious from the proofs of Lemmas 4.2 and 4.3.

Lemma 4.4. *Let S'_0 be as in the proof of Proposition 4.1 (or Lemmas 4.2 and 4.3) and $\mathcal{S}'_1 \to \Delta$ a degeneration of del-Pezzo surfaces introduced in the proof of Lemma 4.2, having S'_0 as the central fiber. Then there exists (in general reducible) 56 curves on S'_0 satisfying the following properties: (i) the self-intersection numbers of the 56 curves on S'_0 are (-1), (ii) the 56 curves can be naturally extended to (-1)-curves on S_t for any $t \in \Delta$, $t \neq 0$, (iii) the (-1)-curves obtained in (ii) are the set of (-1)-curves on S_t.*

So far we have considered the simplest situation that the quartic degenerates to have a unique node. From now on we consider the situation it actually happens for our quartic surface. Recall that if $H \in \langle l_\infty \rangle^\sigma$ (and $H \neq H_i, H_{\lambda_0}$), then $H \cap B$ is a union of two $\mathbf{C}^*$-invariant conics and that the two conics touches each other at two $\mathbf{C}^*$-fixed points. So let B_0 be such a $\mathbf{C}^*$-invariant quartic. B_0 has just two singular points, both of which are A_3-singularities of a curve.

Proposition 4.5. *B_0 has three bitangents. One is the line connecting the two singular points of B_0 and the other two are the two common tangents at the singular points of B_0.*

Proof. It suffices to show that there are not bitangents other than the three bitangents in the proposition. So assume that l is such a bitangent of B_0. Then it is immediate to see that l does not pass through the two singular points of B_0. This implies that l is not $\mathbf{C}^*$-invariant. Moreover, being bitangent is an invariant condition under $\mathbf{C}^*$-action since the branch curve B_0 is assumed to be $\mathbf{C}^*$-invariant. Hence we get a one-dimensional family of bitangents of B_0. However, bitangents always must be isolated, since the inverse image contains (-1)-curve as an irreducible component, which cannot be moved in the double covering. Hence there are no bitangents other than the three ones. $\square$

In the following we investigate the behavior of 28 bitangents when a smooth plane quartic degenerates into the above B_0. We consider a family of plane quartics $\mathcal{B}_2 \to \Delta$ such that B_t (= the fiber over t) is smooth for $t \neq 0$ and the central fiber is B_0. The following lemma corresponds to Lemma 4.2 in the simplest case. In the subsequent statement l_1 means the line through the two (A_3-) singularities of B_0, and l_2 and l_3 mean the remaining two bitangents (which are common tangents of the two irreducible components of B_0).

Lemma 4.6. *The 28 bitangents of B_t, $t \neq 0$, can be grouped into three subsets $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ which behave as follows when t goes to 0 in Δ: every bitangent in $\mathcal{G}_i$ converges to l_i for $i = 1, 2$ and 3. Furthermore, $\mathcal{G}_1$ consists of 16 bitangents, whereas both of $\mathcal{G}_2$ and $\mathcal{G}_3$ consist of 6 bitangents.*

Proof. Take a double cover $\mathcal{S}_2 \to \mathbf{CP}^2 \times \Delta$ branching along $\mathcal{B}_2$. Then $\mathcal{S}_2$ has two compound A_3-singularities, which are of course over the singular points of B_0. By using the natural projection $\mathcal{S}_2 \to \Delta$, $\mathcal{S}_2$ can be viewed as a degeneration of del-Pezzo surfaces (of degree two) into a surface S_0 which has two A_3-singularities. (S_0 is the double cover branched along B_0.) Hence by a well-known property of rational double points of surface, $\mathcal{S}_2 \to \Delta$ admits a simultaneous resolution $\mathcal{S}_2' \to \mathcal{S}_2$, which can be also regarded a small resolution of the compound singularities of $\mathcal{S}_2$. (There are many small resolutions of $\mathcal{S}_2$ as will be explicitly given in Section 7.3.) It is also well know that over the central fiber the simultaneous resolutions give the minimal resolution of the A_3-singularities of S_0. We again denote $\psi : S_0' \to \mathbf{CP}^2$ for the composition of the minimal resolution with the covering map $S_0 \to \mathbf{CP}^2$ as in the proof of Lemma 4.2. The exceptional curve of the minimal resolution of a A_3-singularity is a chain of three (-2)-curves which we write $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where we may suppose $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_3 = 1$ and $\Gamma_3\Gamma_1 = 0$ on S_0'. We denote the exceptional curve over the other A_3-singularity by $\overline{\Gamma} = \overline{\Gamma}_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$ (since we keep in mind the case that B_0 and S_0 have real structures). These are also (-2)-curves on S_0' satisfying $\overline{\Gamma}_1\overline{\Gamma}_2 = \overline{\Gamma}_2\overline{\Gamma}_3 = 1$ and $\overline{\Gamma}_3\overline{\Gamma}_1 = 0$. l_1 was the line connecting the two A_3-singularities of B_0. Then $\psi^{-1}(l_1)$ is a cycle of 8 smooth rational curves, six of which are $\Gamma_1, \cdots, \overline{\Gamma}_3$ and the other two are smooth rational curves which are mapped biholomorphically onto l_1. We still denoted these curves by Ξ and $\overline{\Xi}$. (See Figure 5.) We claim that $\Xi^2 = \overline{\Xi}^2 = -1$ on S_0'. Since we still have $-K = \psi^*O(1)$ for the anticanonical class of S_0', and since we have $c_1^2(S_0') = 2$, we have $(\psi^{-1}(l_1))^2 = 2$ on S_0'. Because the incidence relations of the irreducible components of $\psi^{-1}(l_1)$ are as in Figure 5, we easily get $\Xi^2 + \overline{\Xi}^2 = -2$. Suppose that one of Ξ and $\overline{\Xi}$, say Ξ satisfies $\Xi^2 \geq 0$ on S_0'. Then by deformation theory, Ξ has a non-trivial deformation in S_0'. By taking the image by ψ, we get a non-trivial family of lines. This cannot happen because the curves obtained as deformation of Ξ in S_0' always intersect Γ and $\overline{\Gamma}$ (because $\Gamma\Xi = \overline{\Gamma}\Xi = 1$) and therefore the image (by ψ) must go through the two singular points of B_0. Thus we get the claim $\Xi^2 = \overline{\Xi}^2 = -1$ on S_0'. Then as in the proof of Lemma 4.3, a connected reduced curve $\Omega \subset \psi^{-1}(l_1)$ containing Ξ but not containing $\overline{\Xi}$ as its irreducible component will uniquely extended to be a (-1)-curve on S_t for small $t \in \Delta$. It is easy to

list up all such curves $\Omega \subset \psi^{-1}(l_1)$ in explicit forms and conclude that the number of such Ω is $4 \times 4 = 16$. Moreover, the intersection numbers of any two different curves among these 16 curves can be verified to be 0 or 1. It then follows, as in the final part of the proof of Lemma 4.3, that any of the 16 (-1)-curves on S_t $(t \neq 0)$ are mapped (by $\phi_t : S_t \to \mathbf{CP}^2$) to mutually different bitangents. By construction, all these 16 bitangents clearly converge to l_1 when $t \to 0$. These bitangents of B_t give the members of $\mathcal{G}_1$ in the lemma.

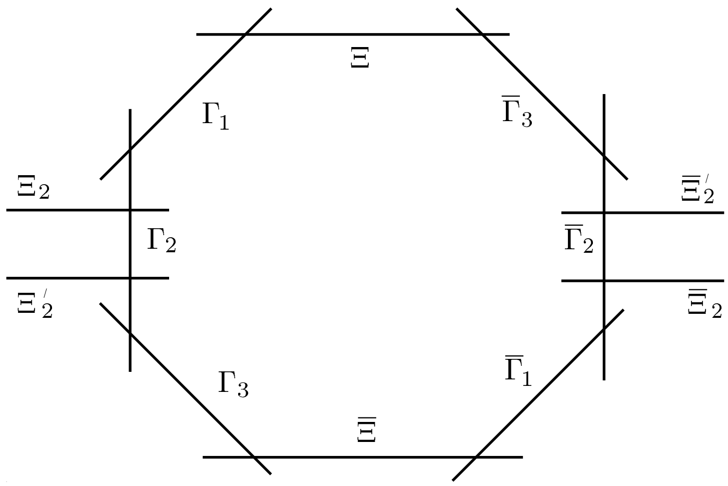

FIGURE 5. the inverse image of $l_1 + l_2 + l_3$

Next we look for 6 bitangents in $\mathcal{G}_2$ in a similar way. $\psi^{-1}(l_2)$ consists of 5 smooth rational curves, three of which are Γ_1, Γ_2 and Γ_3, and the remaining two are mapped biholomorphically onto l_2. Let Ξ_2 and Ξ_2' be the latter two components, which are (-1)-curves on S_0'. In this case, one should note that $\psi^{-1}(l_2)$ is non-reduced and contains Γ_2 as a component of multiplicity two. Moreover, we have $\Xi_2\Gamma_1 = \Xi_2\Gamma_3 = \Xi_2'\Gamma_1 = \Xi_2'\Gamma_3 = 0$ and $\Xi_2\Gamma_2 = \Xi_2'\Gamma_2 = 1$ on S_0' (see Figure 5). (These can be easily seen by calculations using local coordinates. We do not explain it here since in Lemma 7.19, we calculate the intersection numbers of Γ_i's and the inverse image of a conic which has l_2 and l_3 as a tangent line at the two singular points of B_0.) Then we take 6 curves

$$\Xi_2, \ \ \Xi_2 + \Gamma_2, \ \ \Xi_2 + \Gamma_2 + \Gamma_1, \ \ \Xi_2 + \Gamma_2 + \Gamma_3, \ \ \Xi_2 + \Gamma_2 + \Gamma_1 + \Gamma_3, \ \ \Xi_2 + 2\Gamma_2 + \Gamma_1 + \Gamma_3 \tag{9}$$

as candidates of curves which extend to be (-1)-curves on S_t, $t \neq 0$. It is easily verified that the self-intersections of these 6 curves are -1. Then again an argument in the proof of Lemma 4.3 with a slight modification shows that these curves (9) also can be deformed to be (-1)-curves on S_t. Moreover the intersection numbers of the different two curves among (9) are either 0 or 1 and not 2. Hence the images of the (-1)-curves on S_t must be mutually different. Thus we get 6 bitangents whose limit as $t \to 0$ are l_2. These give the members of $\mathcal{G}_2$ in the lemma.

Finally, the remaining 6 bitangents in $\mathcal{G}_3$ are similarly obtained by considering l_3, instead of l_2 in the above argument. □

Analogous to Lemma 4.4, we have the following

Lemma 4.7. *Let $\mathcal{S}_2' \to \Delta$ be the degeneration of del-Pezzo surfaces of degree two introduced in the proof of Lemma 4.6 and S_0' the fiber over the origin. Then there exist (generally reducible) 56 curves on S_0' satisfying the following properties: (i) the self-intersection numbers of the 56 curves on S_0' are (-1), (ii) the 56 curves can be naturally extended to (-1)-curves on S_t for $t \in \Delta$, $t \neq 0$, (iii) the (-1)-curves obtained in (ii) are the set of (-1)-curves on S_t.*

Proof. We continue to use the notations in the proof of Lemma 4.6. In the proof, we have obtained sixteen connected curves contained in $\psi^{-1}(l_1)$ containing Ξ but not containing $\overline{\Xi}$ as their irreducible components. These give sixteen (-1)-curves on S_t, $t \neq 0$. Interchanging the role of Ξ and $\overline{\Xi}$, we obtain other sixteen (-1)-curves on S_t. Thus we get $16 \times 2 = 32$ (-1)-curves so far. On the other hand, the six curves (9) yield six (-1)-curves on S_t. Exchanging Ξ_2 and Ξ_3, we get other six (-1)-curves on S_t. Thus we get $6 \times 2 = 12$ (-1)-curves on S_t from the curves contained in $\psi^{-1}(l_2)$. Then by exchanging the role of l_2 and l_3, we get twelve (-1)-curves. Thus we get $32 + 2 \times 12 = 56$ (-1)-curves on S_t. These must be all of the (-1)-curves on S_t by Proposition 3.7. $\square$

Next we take real structures into consideration and prove Proposition 3.14 (non-existence of non-trivial real bitangent). Let $\mathcal{S}_2 \to \Delta$ be the degeneration of del-Pezzo surface constructed from $\mathcal{B}_2 \to \Delta$ as above and suppose that $\mathcal{B}_2$ is invariant under the real structure on $\mathbf{CP}^2 \times \Delta$ which is the product of standard real structures on $\mathbf{CP}^2$ and $\Delta \subset \mathbf{C}$ (i.e. complex conjugations). Then $\mathcal{S}_2$ admits a natural real structure and a fiber S_t is real if $t \in \Delta^\sigma$. Suppose moreover that on the real fibers of $\mathbf{CP}^2 \times \Delta \to \Delta$ the real structure keeps the line l_1 invariant and exchanges l_2 and l_3. Then the two A_3-singularities of B_0 are necessarily a conjugate pair and $l_2 \cap l_3$ is a real point. Moreover there are simultaneous resolutions $\mathcal{S}_2' \to \Delta$ of $\mathcal{S}_2 \to \Delta$ such that the real structure on $\mathcal{S}_2$ lifts onto $\mathcal{S}_2'$, since once one of the singularities is resolved, the other singularity is automatically resolved by reality. We still denote by $\psi : S_0' \to \mathbf{CP}^2$ for the composition of the minimal resolution $S_0' \to S_0$ and the covering map $S_0 \to \mathbf{CP}^2$ whose branch is B_0.

Lemma 4.8. *Let $\mathcal{B}_2 \to \Delta$ and $\mathcal{S}_2' \to \mathcal{S}_2 \to \Delta$ be as above and suppose that the real structure acts on $\psi^{-1}(l_1)$ in such a way that it interchanges irreducible components on the opposite side. Then for $t \in \Delta^\sigma$ and $t \neq 0$ the number of real bitangents of B_t is four.*

Proof. By Lemma 4.7, 28 bitangents of B_t ($t \neq 0$) are obtained from (generally reducible) 56 curves on S_0' by smoothing and then taking the image by the double covering map. Two members E_1 and E_2 among the 56 curves on S_0' generate (-1)-curves that are mapped the same bitangent of B_t precisely when $E_1E_2 = 2$ on S_0'. Thus it suffices to show that there are just four such pairs $\{E_1, E_2\}$ satisfying $E_2 = \overline{E}_1$. This is possible because the 56 curves have been explicitly determined. Actually, if E_1 is contained $\psi^{-1}(l_2)$ or $\psi^{-1}(l_3)$, then $\overline{E}_1$ is contained in $\psi^{-1}(l_3)$ or $\psi^{-1}(l_2)$ respectively by our assumption on the real structure. Thus we always have $E_1\overline{E}_1 = 0$. So we can suppose that E_1 is contained in $\psi^{-1}(l_1)$. In this case, $E_2 = \psi^{-1}(l_1) - E_1$ is the unique curve (among the 56 curves on S_0') which satisfies $E_1E_2 = 2$. It is easily seen from the action of the real structure on the irreducible components on $\psi^{-1}(l_1)$ that there are just four E_1's satisfying $\overline{E}_1 = \psi^{-1}(l_1) - E_1$. Thus we have obtained the claim of the proposition. $\square$

Proof of Proposition 3.14. It is readily seen by Lemma 3.4 that real lines on $\mathbf{CP}^3$ which are not contained in some $H \in U^\sigma$ are l_∞ and real lines going through P_0. l_∞ is a trivial bitangent and real lines through P_0 cannot be a bitangent as we will see in the proof of Proposition 8.1 (easy to show). Thus to prove the non-existence of non-trivial real bitangent of B, it suffices to show that any $H \in U^\sigma$ has just four real bitangents (which are of course trivial bitangents).

Fix any real plane $H_0 \in \langle l_\infty \rangle^\sigma$ which is different from H_i ($1 \leq i \leq 4$) and H_{λ_0}. Then $H \cap B$ is a union of two $\mathbf{C}^*$-invariant conics. Next take a real line γ in $(\mathbf{CP}^3)^\vee$ going through $H_0 \in (\mathbf{CP}^3)^\vee$. Then γ defines a real pencil of planes containing H_0 as a (real) member, and by considering their intersections with B, we get a one-dimensional family of plane quartics. By taking γ sufficiently general, we can suppose that general members of this family is smooth. Let $\mathcal{B} \to \gamma$ be a family of plane quartics thus obtained. By construction, this family enjoys the same properties as $\mathcal{B}_2 \to \Delta$ in Lemma 4.8. Therefore fibers of $\mathcal{B} \to \gamma$

has just four real bitangents, at least in a neighborhood of $H_0 \in \gamma$. But since we know that U^σ is connected, we get that $H \cap B$ has just four real bitangents for any $H \in U^\sigma$. On the other hand, $H \cap H_i$ $(1 \le i \le 4)$ are actually bitangents of $B \cap H$. This implies that every bitangent of $B \cap H$ is trivial, and we get the claim of Proposition 3.14. $\square$

5. Defining equations of real touching conics

In Section 3 we have shown that real lines in our threefold Z are mapped biholomorphically (by $\Phi : Z \to \mathbf{CP}^3$) onto real touching conics in general (Proposition 3.2). Then we studied touching conics lying on general real planes (i.e. planes in U^σ) and showed that they form 63 one-dimensional families (Proposition 3.10). We also explained that in order to select out the right family which will actually be the image of twistor lines, it is important to investigate real lines contained in $S_H = \Phi^{-1}(H)$ where H is a real $\mathbf{C}^*$-invariant planes (i.e. $H \in \langle l_\infty \rangle^\sigma$), or equivalently, real touching conics lying on these planes. In Sections 5–7 we investigate these special real lines and touching conics. These lines and conics can be viewed as a limit of general twistor lines and their images. Since the plane sections of B by the $\mathbf{C}^*$-invariant planes are singular (splitting into two $\mathbf{C}^*$-invariant conics in general), we cannot apply the results in Section 3. In this section we determine real touching conics lying on these real $\mathbf{C}^*$-invariant planes. Thanks to the simple form of the quartics, it is possible to classify them and write down their defining equation.

Let $B, P_0, P_\infty, \overline{P}_\infty, \Phi_0 : Z_0 \to \mathbf{CP}^3, \mu : Z \to Z_0, \Phi : Z \to \mathbf{CP}^3, \Gamma_0, \Gamma, \overline{\Gamma}$ and σ have the meaning as in the beginning of Section 3. We fix a $\mathbf{C}^*$-action defined by

$$(y_0, y_1, y_2, y_3) \mapsto (y_0, y_1, ty_2, t^{-1}y_3), \quad t \in \mathbf{C}^*$$

(cf. Proposition 2.1 (iii)). Then $\mathbf{C}^*$-invariant real planes must be of the form $H_\lambda = \{y_0 = \lambda y_1\}$, $\lambda \in \mathbf{R}$ or $H_\infty = \{y_1 = 0\}$. (H_{λ_0} is the unique plane in $\langle l_\infty \rangle^\sigma$ going through P_0.) Moreover, as in Section 2, we put

$$f(\lambda) = \lambda(\lambda + 1)(\lambda - a).$$

(Note that $a > 0$.) The sign of $f(\lambda)$ will be important in the sequel. We are assuming that Q and f satisfy Condition (A) in Proposition 2.6.

As seen in the proof of Proposition 2.5, $B_\lambda = B \cap H_\lambda$ is a union of two $\mathbf{C}^*$-invariant conics, and their intersection is P_∞ and $\overline{P}_\infty$. Note that if $f(\lambda) = 0$ (namely if $\lambda = -1, 0$ or a) or $\lambda = \infty$, B_λ becomes a trope. Let C be a real touching conic contained in H_λ, $\lambda \neq -1, 0, a, \infty$. Since the two components of B_λ have the same tangent lines at P_∞ and $\overline{P}_\infty$, the local intersection number $(C, B_\lambda)_{P_\infty}$ is zero, two, or four, and by reality, the same holds for $(C, B_\lambda)_{\overline{P}_\infty}$. Correspondingly, real touching conics lying on some H_λ can be classified into the following three types:

Definition 5.1. Let C be a real touching conic in H_λ, $\lambda \neq -1, 0, a, \infty$. Then (i) C is called *generic type* if $(C, B_\lambda)_{P_\infty} = (C, B_\lambda)_{\overline{P}_\infty} = 0$. (ii) C is called *special type* if $(C, B_\lambda)_{P_\infty} = (C, B_\lambda)_{\overline{P}_\infty} = 2$. (iii) C is called *orbit type* if $(C, B_\lambda)_{P_\infty} = (C, B_\lambda)_{\overline{P}_\infty} = 4$. (See Figure 6.)

If $C \subset H_\lambda$ is a real touching conic of generic type, then $P_\infty, \overline{P}_\infty \notin C$ and $C \cap B_\lambda$ consists of just four points, all satisfying $(C, B_\lambda)_P = 2$. If C is a real touching conic of special type, then C goes through P_∞ and $\overline{P}_\infty$ but the tangent lines at P_∞ and $\overline{P}_\infty$ are different from the common bitangents of B_λ. Further, there are other intersection P and $\overline{P}$ satisfying $(C, B_\lambda)_P = (C, B_\lambda)_{\overline{P}} = 2$. If C is a real touching conic of orbit type, there are no other intersection points. In this case, C is the closure of $\mathbf{C}^*$-action, where $\mathbf{C}^*$-action is the complexification of $U(1)$-action.

First we classify real touching conics of generic type:

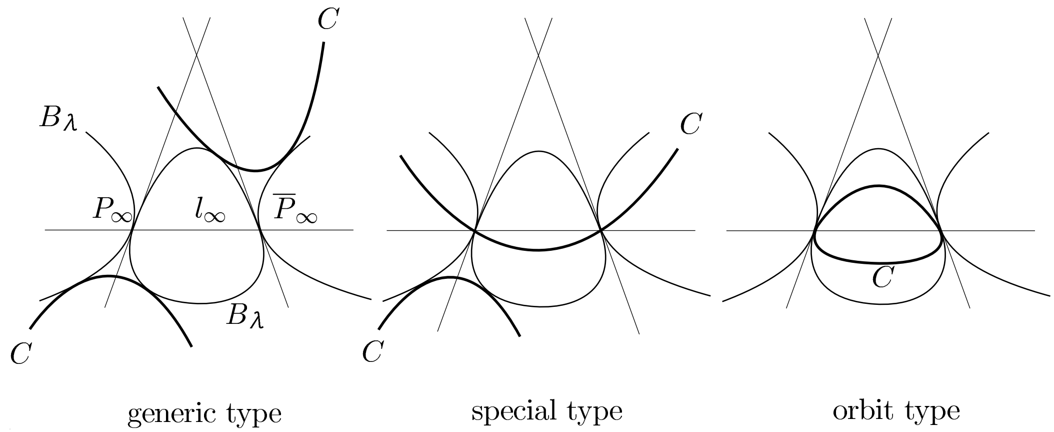

FIGURE 6. real touching conics (C is irreducible)

Proposition 5.2. *(i) If $\lambda \in \mathbf{R}$ satisfies $f(\lambda) > 0$ and $Q(\lambda, 1)^2 > f(\lambda)$ (i.e. $\lambda \neq \lambda_0$), there exists a family of real touching conics of generic type on H_λ, parametrized by a circle. Their defining equations are explicitly given by*

$$2(Q^2 - f)y_1^2 + \sqrt{f}e^{i\theta}y_2^2 + 2Qy_2y_3 + \sqrt{f}e^{-i\theta}y_3^2 = 0, \tag{10}$$

where we put $Q = Q(\lambda, 1)$ and $f = f(\lambda)$, and $\theta \in \mathbf{R}$. Further, every real touching conic of generic type in H_λ is a member of this family. (ii) If $f(\lambda) < 0$ or $Q(\lambda, 1)^2 = f(\lambda)$, there is no real touching conic of generic type on H_λ. (iii) For the case (i), the conic (10) has no real point for any $\theta \in \mathbf{R}$.

We note that $U(1)$ acts transitively (but non-effectively) on the space of these touching conics. To prove the proposition, we need the following

Lemma 5.3. *An equation $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ has two double roots iff (i) if $a_1 \neq 0$, then $4a_1a_2 = a_1^3 + 8a_3$ and $a_1^2a_4 = a_3^2$ hold, (ii) if $a_1 = 0$, then $a_3 = 0$ and $4a_4 = a_2^2$ hold. In these cases, the double roots are given by*

$$-\frac{a_1}{4} \pm \sqrt{\frac{a_1^2}{16} - \frac{a_3}{a_1}}$$

for the case (i) and

$$\pm\sqrt{\frac{-a_2}{2}}$$

for the case (ii).

We omit a proof since it is elementary.

Proof of Proposition 5.2. Let $ay_1^2 + by_1y_2 + cy_2^2 + dy_1y_3 + ey_2y_3 + hy_3^2 = 0$ be a defining equation of C in H_λ. Then since $B_\lambda \cap \{y_3 = 0\} = \{\overline{P}_\infty\}$ and since C is assumed to be generic type, all of the touching points are on $\{y_3 \neq 0\}$. Putting $x_1 = y_1/y_3$ and $x_2 = y_2/y_3$ on $\{y_3 \neq 0\}$ as before, C is defined by

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + h = 0 \tag{11}$$

and B_λ is defined by (as in the proof of Proposition 2.5)

$$\left(x_2 + \left(Q - \sqrt{f}\right)x_1^2\right)\left(x_2 + \left(Q + \sqrt{f}\right)x_1^2\right) = 0.$$

Let g denote $g_+ := Q + \sqrt{f}$ or $g_- := Q - \sqrt{f}$. Substituting $x_2 = -gx_1^2$ into (11), we get

$$g^2cx_1^4 - gbx_1^3 + (a - ge)x_1^2 + dx_1 + h = 0. \tag{12}$$

If $c=0$, (12) cannot have two double roots, so we have $c \neq 0$. Suppose $b \neq 0$. Then by Lemma 5.3 (i), (12) has two double roots iff

$$-\frac{4b}{cg}\cdot\frac{a-eg}{cg^2} = -\frac{b^3}{c^3g^3}+\frac{8d}{cg^2} \quad \text{and} \quad \left(-\frac{b}{cg}\right)^2\cdot\frac{h}{cg^2} = \frac{d^2}{c^2g^4}$$

hold. These can be respectively written

$$4bc(a-eg) = b^3 - 8gc^2d, \quad b^2h = cd^2. \tag{13}$$

Namely, a conic (11) with $b \neq 0$ is a touching conic of generic type iff (13) is satisfied for both of $g = g_+$ and $g = g_-$. In this case, simple calculations show that $4ac = b^2$ and $4ah = d^2$ and $2ae = bd$. From these we get $a = b^2/4c, e = 2cd/b$ and $h = cd^2/b^2$. Substituting these into (11), we get $(bx_1 + 2cx_2 + 2cd/b)^2 = 0$. This implies that C is a double line. Thus contradicting our assumption and we get $b = 0$. Then by Lemma 5.3 (ii), we have $d = 0$ and

$$4g^2ch = (a-ge)^2. \tag{14}$$

If we regard (14) as a homogeneous equation about $(a:c:e:h) \in \mathbf{CP}^3$, (14) is a quadratic cone whose vertex is $(a:c:e:h) = (g:0:1:0)$. We need to get the intersection of these two quadrics. Restricting (14) onto the plane $a = \kappa e$, we get

$$4g^2ch = (\kappa-g)^2e^2, \ \ g = g_{\pm}. \tag{15}$$

It is readily seen that these two conics (for the case $g = g_+$ and $g = g_-$) coincide iff $\kappa = 0$ or $\kappa = (Q^2-f)/Q$. If $\kappa = 0$, we have $a = 0$, so (11) becomes $cx_2^2 + ex_2 + h = 0$, where the coefficients are subjected to $4ch = e^2$. This implies that C is a union of two lines, contradicting our assumption. Hence we have $\kappa = (Q^2-f)/Q$. Then (15) becomes

$$4Q^2ch = fe^2. \tag{16}$$

If $e = 0$, it follows that $h = 0$, and hence (11) will again be a union of lines. Therefore we have $e \neq 0$. It is easily seen that the real structure on the space of coefficients is given by $(a:c:e:h) \mapsto (\overline{a}:\overline{h}:\overline{e}:\overline{c})$. Hence if $f = f(\lambda) < 0$, (16) cannot hold for real $(a:c:e:h)$. Namely, on H_λ, there does not exist a real touching conic of generic type if $f(\lambda) < 0$. This implies (ii) of the proposition for the case $f(\lambda) < 0$. If $f = f(\lambda) > 0$, putting $e = 1$ and $h = \overline{c}$ in (16) we get $4Q^2|c|^2 = f$. Hence we can write

$$c = \frac{\sqrt{f}}{2Q}e^{i\theta}$$

for some $\theta \in \mathbf{R}$. Further we have $a = (Q^2-f)/Q$. Substituting these into (11), we get (10). Then it is immediate to see that the determinant of the matrix defining (10) is $-2(Q^2-f)^2$. Therefore the conic (10) is irreducible iff $Q^2 - f \neq 0$. Thus we get (i), and also (ii) for the case $Q(\lambda)^2 = f(\lambda)$.

Finally we show (iii). Recall that the real structure on H_λ is given by $(y_1:y_2:y_3) \mapsto (\overline{y}_1:\overline{y}_3:\overline{y}_2)$. Hence if $(y_1:y_2:y_3) \in H_\lambda$ is a real point, we can suppose $y_1 \in \mathbf{R}$ and $y_3 = \overline{y}_2$. Substituting these into (10), we get

$$(Q^2-f)y_1^2 + Q|y_2|^2 + \sqrt{f}\cdot\mathrm{Re}(e^{i\theta}y_2^2) = 0, \tag{17}$$

where $\mathrm{Re}(z)$ denotes the real part of z. From this it follows that $y_2 = 0$ implies $y_3 = y_1 = 0$, so we have $y_2 \neq 0$. Then recalling that $Q > \sqrt{f}$ (Proposition 2.5 (i)) we have $Q|y_2|^2 > \sqrt{f}|y_2|^2$. Also we have $\mathrm{Re}(e^{i\theta}y_2^2) \leq |y_2|^2$. Therefore we have

$$\begin{aligned} (Q^2-f)y_1^2 + Q|y_2|^2 + \sqrt{f}\cdot\mathrm{Re}(e^{i\theta}y_2^2) &> (Q^2-f)y_1^2 + \sqrt{f}|y_2|^2 - \sqrt{f}|y_2|^2 \\ &= (Q^2-f)y_1^2 \ \geq 0 \end{aligned}$$

This implies that (17) does not hold for any real $(y_1:y_2:y_3)$ and $\theta \in \mathbf{R}$. Thus there is no real point on the conic (10) provided $Q^2 > f > 0$, and we get (iii). $\square$

Next we classify real touching conics of special type:

Proposition 5.4. *(i) If $\lambda \in \mathbf{R}$ satisfies $f(\lambda) > 0$, there is no real touching conic of special type on H_λ. (ii) If $f(\lambda) < 0$, there exists a family of real touching conics of special type, parametrized by a circle. Their defining equations are given by*

$$\sqrt{Q^2 - f} \cdot y_1^2 + \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}} \cdot e^{i\theta} y_1 y_2 + \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}} \cdot e^{-i\theta} y_1 y_3 + y_2 y_3 = 0, \tag{18}$$

where we put $Q = Q(\lambda, 1)$ and $f = f(\lambda)$, and $\theta \in \mathbf{R}$ as before. Further, every real touching conic of special type in H_λ is a member of this family. (iii) The conic (18) has no real point for any $\theta \in \mathbf{R}$.

Note again that $U(1)$ acts transitively on the parameter space of these touching conics. Also note that if $f < 0$ we have $Q^2 - f > 0$ and $\sqrt{Q^2 - f} - Q > 0$. Hence every square root in the equation make a unique sense (i.e. we always take the positive root).

Proof. Let C be a real touching conic of special type on H_λ. Then since C goes through P_∞ and $\overline{P}_\infty$, the other two touching points belong to mutually different irreducible component of B_λ. On the other hand, as shown in the proof of Proposition 2.5, each irreducible components of B_λ is real iff $f(\lambda) \geq 0$. Therefore, on H_λ, there does not exist real touching conic of special type if $f(\lambda) > 0$. Thus we get (i). So in the sequel we suppose $f(\lambda) < 0$.

It is again readily seen that touching points are not on the line $\{y_3 = 0\}$. So we still use (x_1, x_2) as a non-homogeneous coordinate on $\mathbf{C}^2 = \{y_3 \neq 0\} \subset H_\lambda$. Then because C contains P_∞ and $\overline{P}_\infty$, an equation of a touching conic C of special type is of the form

$$a x_1^2 + b x_1 x_2 + d x_1 + e x_2 = 0. \tag{19}$$

Substituting $x_2 = -g x_1^2$ into (19), we get

$$x_1 \cdot \left(g b x_1^2 + (ge - a) x_1 - d \right) = 0. \tag{20}$$

If $d = 0$, $x_1 = 0$ is a double root of (20). Then the tangent line of C at $P_\infty = (0, 0)$ becomes $x_2 = 0$, as is obvious from (19). This implies that C is a touching conic of orbit type, contradicting our assumption. Therefore, we have $d \neq 0$. (20) has a double root other than $x_1 = 0$ iff

$$(ge - a)^2 + 4gbd = 0. \tag{21}$$

Namely (19) is a touching conic of special type iff (21) is satisfied for both of $g = g_+$ and $g = g_-$. (Note that $g_+ \neq g_-$.) If we regard (21) as a homogeneous equation of $(a : b : d : e) \in \mathbf{CP}^3$, (21) is a quadratic cone whose vertex is $(a : c : d : e) = (g : 0 : 0 : 1)$. That is, the parameter space of touching conics of special type is the intersection of two quadratic cones in $\mathbf{CP}^3$. Restricting (21) onto the plane $a = \kappa e$, we get

$$(g - \kappa)^2 e^2 + 4gbd = 0, \ g = g_\pm.$$

It is readily seen that these two conics coincide iff $\kappa = \pm\sqrt{g_+ g_-} = \pm\sqrt{Q^2 - f}$. Therefore C is a touching conic of special type iff

$$a = \sqrt{Q^2 - f} \cdot e, \ \left(Q - \sqrt{Q^2 - f} \right) e^2 + 2bd = 0 \tag{22}$$

or

$$a = -\sqrt{Q^2 - f} \cdot e, \ \left(Q + \sqrt{Q^2 - f} \right) e^2 + 2bd = 0 \tag{23}$$

hold. It is easily seen that the real structure on the space of coefficients is given by $(a : b : d : e) \mapsto (\overline{a} : \overline{d} : \overline{b} : \overline{e})$.

Since we have assumed $f < 0$, we have $Q + \sqrt{Q^2 - f} > 0$ and there is no real conic satisfying (23). On the other hand, we have $Q - \sqrt{Q^2 - f} < 0$. Hence by (22) we have

$$2|b|^2 = \left(\sqrt{Q^2 - f} - Q\right) e^2,$$

where $b \in \mathbf{C}$ and $e \in \mathbf{R}$. If $e = 0$, then $b = a = 0$, contradicting the assumption that C is a conic. Hence $e \neq 0$, and we may put $e = 1$. Then we can write

$$b = \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}} \cdot e^{i\theta}$$

for some $\theta \in \mathbf{R}$. Also we have $d = \overline{b}$. Thus we obtain (18) of the proposition.

Finally we show (iii). If $(y_1 : y_2 : y_3)$ is a real point of H_λ, we can suppose $y_1 \in \mathbf{R}$ and $y_3 = \overline{y}_2$. Substituting these into (18), we get

$$\sqrt{Q^2 - f} \cdot y_1^2 + \sqrt{2\left(\sqrt{Q^2 - f} - Q\right)} \cdot y_1 \cdot \mathrm{Re}(e^{i\theta} y_2) + |y_2|^2 = 0. \tag{24}$$

If $y_1 = 0$, it follows $y_2 = y_3 = 0$. Hence $y_1 \neq 0$ and we can suppose $y_1 > 0$. Then we have $y_1 \mathrm{Re}(e^{i\theta} y_2) \geq -y_1|y_2|$. Hence we have

$$\begin{aligned}
&\sqrt{Q^2 - f} \cdot y_1^2 + \sqrt{2\left(\sqrt{Q^2 - f} - Q\right)} \cdot y_1 \mathrm{Re}(e^{i\theta} y_2) + |y_2|^2 \\
\geq\ & \sqrt{Q^2 - f} \cdot y_1^2 - \sqrt{2\left(\sqrt{Q^2 - f} - Q\right)} \cdot y_1 |y_2| + |y_2|^2 \\
=\ & \left(|y_2| - \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}} \cdot y_1\right)^2 + \frac{\sqrt{Q^2 - f} + Q}{2} \cdot y_1^2
\end{aligned}$$

Because $y_1 \neq 0$ and $f < 0$, we have $(\sqrt{Q^2 - f} + Q) y_1^2 > 0$. Therefore, the left hand side of (24) is strictly positive. Thus (24) does not hold for any real $(y_1 : y_2 : y_3) \in H_\lambda$ and any $\theta \in \mathbf{R}$. Therefore the conic (18) has no real point for any $\theta \in \mathbf{R}$.

It is immediate to see that the determinant of the matrix defining (18) is $-(Q + \sqrt{Q^2 - f})/8$ and this is negative if $f < 0$. Hence the conic (18) is irreducible □

The case of orbit type is straightforward and need no assumption on the sign of $f(\lambda)$:

Proposition 5.5. *There exists a family of real touching conics of orbit type, parametrized by non-zero real numbers. Their defining equations are*

$$y_2 y_3 = \alpha y_1^2, \ \alpha \in \mathbf{R}^\times. \tag{25}$$

Further, every real touching conic of orbit type in H_λ is contained in this family.

Note that by Lemma 2.4, the conic (25) has no real point iff $\alpha < 0$.

Combining Propositions 5.2, 5.4 and 5.5, we get the following

Proposition 5.6. *Let $\{S_\lambda := \Phi^{-1}(H_\lambda) \,|\, H_\lambda \in \langle l_\infty \rangle^\sigma\}$ be the real members of the pencil of $U(1)$-invariant divisors on Z. Then (i) if $f(\lambda) > 0$, the images of real lines in S_λ are real touching conics of generic type or orbit type, (ii) if $f(\lambda) < 0$, the images of real lines in S_λ are real touching conics of special type or orbit type.*

In Section 7 we will determine which one of the two candidates in the proposition must be the images of twistor lines (Proposition 7.23).

6. The inverse images of real touching conics

According to the previous section, real touching conics in a $\mathbf{C}^*$-invariant real plane form families parametrized by a circle for generic and special types, or $\mathbf{R}^{\times}(\ni \alpha)$ for orbit type. In this section we study the inverse images in Z of these touching conics, which are candidates of twistor lines. We continue to use the same notations and assumptions. Recall that if $\lambda \neq -1, 0, a, \infty$, $S_\lambda = \Phi^{-1}(H_\lambda)$ is a smooth rational surface by Proposition 3.5. Clearly S_λ is $\mathbf{C}^*$-invariant real member of $|(-1/2)K_Z|$ (Proposition 2.8).

First we investigate the inverse images of touching conics of generic type.

Proposition 6.1. *Suppose that $\lambda \in \mathbf{R}$ satisfies $f(\lambda) > 0$ and $\lambda \neq \lambda_0$ (namely $Q^2 - f > 0$), and let $C_\theta \subset H_\lambda$ be a real touching conic of generic type defined by the equation (10). Then the following (i)–(iv) hold: (i) $\Phi^{-1}(C_\theta)$ has just two irreducible components, both of which are smooth rational curves that are mapped biholomorphically onto C_θ, (ii) each irreducible component of $\Phi^{-1}(C_\theta)$ has a trivial normal bundle in S_λ, (iii) these two irreducible components of $\Phi^{-1}(C_\theta)$ belong to mutually different pencils on S_λ, (iv) each irreducible component of $\Phi^{-1}(C_\theta)$ is real.*

Proof. Since C_θ and the branch quartic B_λ have the same tangent line at any intersection points, it is obvious that $\Phi_0^{-1}(C_\theta)$ splits into two irreducible components L_1 and L_2 which are mapped biholomorphically onto C_θ. Thus we get (i). For (ii) first note that $\Phi^{-1}(C_\theta) = L_1 + L_2$ belongs to $|-2K|$ of S_λ since we have $\Phi^* O_{H_\lambda}(1) \simeq -K$. Hence we have $(-2K)^2 = (L_1 + L_2)^2 = L_1^2 + L_2^2 + 2L_1L_2$ on S_λ. On the other hand, we have $4c_1^2 = 8$ by Proposition 3.5. Hence we get $L_1^2 + L_2^2 + 2L_1L_2 = 8$. Further, since L_1 and L_2 intersect transversally at four points (over the touching points of C_θ with B_λ), we have $L_1L_2 = 4$. Therefore we get $L_1^2 + L_2^2 = 0$. Moreover, by (10), C_θ actually moves in a holomorphic family of curves on H_λ. Hence we have $L_1^2 \geq 0$ and $L_2^2 \geq 0$. Therefore we get $L_1^2 = L_2^2 = 0$. Namely we have (ii). (iii) immediately follows from (ii), since we have $L_1L_2 = 4$ on S_λ. (iv) is harder than one may think at first glance, since there is no real point on C_θ. First we note that it suffices to prove the claim for C_0 (= the curve obtained by setting $\theta = 0$ for C_θ), since $U(1)$ acts transitively on the parameter space of real touching conics of generic type (see Proposition 5.2). The idea of our proof of the reality is as follows: the map $\Phi^{-1}(C_0) \to C_0$ is finite, two sheeted covering whose branch consists of four points. We choose a real simple closed curve $\mathcal{C}$ in C_0 containing all of these branch points, in such a way that over $\mathcal{C}$ we can distinguish two sheets, so that we can explicitly see the reality of each irreducible components. To this end, we still use $(y_1 : y_2 : y_3)$ as a homogeneous coordinate on H_λ and set $V := \{y_1 \neq 0\}$ ($\simeq \mathbf{C}^2$), which is clearly a real subset of H_λ, and use $(v_2, v_3) = (y_2/y_1, y_3/y_1)$ as an affine coordinate on V. Then $Z_0|_V = \Phi_0^{-1}(V)$ is defined by the equation

$$z^2 + (v_2v_3 + Q)^2 - f = 0, \tag{26}$$

where z is a fiber coordinate of $O(2)$, and $Q = Q(\lambda, 1)$, $f = f(\lambda)$ as in the previous section. The real structure is given by $(v_2, v_3, z) \mapsto (\overline{v}_3, \overline{v}_2, \overline{z})$. Then on V, our equation (10) of C_0 becomes $2(Q^2 - f) + \sqrt{f}v_2^2 + 2Qv_2v_3 + \sqrt{f}v_3^2 = 0$. Now we introduce a new coordinate $(u, v) := (v_2 + v_3, v_2 - v_3)$ which is valid on V. Then our real structure is given by $(u, v) \mapsto (\overline{u}, -\overline{v})$, and the defining equation of C_0 becomes $4(Q^2 - f) + \sqrt{f}(u^2 + v^2) + Q(u^2 - v^2) = 0$. From this we immediately have

$$C_0 : \quad v^2 = 4\left(Q + \sqrt{f}\right) + \frac{Q + \sqrt{f}}{Q - \sqrt{f}}u^2. \tag{27}$$

We put $V' := \{(u, v) \in V \,|\, u \in i\mathbf{R}, v \in \mathbf{R}\}$ which is clearly a real subset of V. Then $\mathcal{C} := V' \cap C_0$ is a real simple closed curve (an ellipse) in $V' \simeq \mathbf{R}^2$. Indeed, putting $u = iw$

($w \in \mathbf{R}$), we get from (27)

$$C_0: \quad v^2 + \frac{Q+\sqrt{f}}{Q-\sqrt{f}} w^2 = 4\left(Q+\sqrt{f}\right).$$

(Note that $Q - \sqrt{f} > 0$ by our assumption $f > 0$ and Proposition 2.6.) Substituting $v_2 v_3 = (u^2 - v^2)/4$ into (26), and then using (27), we get

$$\Phi^{-1}(C_0): \quad 4\left(Q-\sqrt{f}\right)^2 z^2 + f u^2 \left(u^2 + 4\left(Q - \sqrt{f}\right)\right) = 0, \tag{28}$$

or, using w above,

$$\Phi^{-1}(C_0): \quad 4\left(Q-\sqrt{f}\right)^2 z^2 = f w^2 \left(4\left(Q - \sqrt{f}\right) - w^2\right). \tag{29}$$

Here note that in (27) u can be used as a coordinate on C_0, only outside the two branch points of $C_\lambda \cap V \to \mathbf{C}$ defined by $(u, v) \mapsto u$. In a neighborhood of these branch points, we have to use v instead of u as a local coordinate on C_0. Then we can see that the inverse image of a neighborhood of $u = \pm 2i(Q - \sqrt{f})^{\frac{1}{2}}$ (i.e. the branch points) also splits into two irreducible components, which is of course as expected. From (28), we easily deduce that the branch points of $\Phi^{-1}(C_0) \to C_0$ are $(u, v) = (0, \pm 2(Q + \sqrt{f})^{\frac{1}{2}})$ and $(u, v) = (\pm 2i(Q - \sqrt{f})^{\frac{1}{2}}, 0)$. All of these four points clearly lie on $\mathcal{C}$, and $\mathcal{C}$ is divided into four segments. It immediately follows from (29) that z always takes real value over $\mathcal{C}$. Moreover, it is clear that the sign of z is constant on each of the four segments in $\mathcal{C}$, and that the sign changes when passing though the branch points. On the other hand, since the real structure is given by $(w, v) \mapsto (-w, -v)$ on V', the real structure on $\mathcal{C}$ sends each segment to another segment which is not adjacent to the original one. From these, and because the real structure on $\Phi^{-1}(\mathcal{C})$ is given by $(w, z) \mapsto (-w, \overline{z}) = (-w, z)$, it follows that each of the two irreducible component of $\Phi^{-1}(\mathcal{C})$ is real. Hence the same is true for $\Phi^{-1}(C_0)$. Thus we have proved (iv) of the proposition. □

We have similar statements for touching conics of special type:

Proposition 6.2. *Assume $f(\lambda) < 0$ and let $C_\theta \subset H_\lambda$ be a real touching conic of special type given by the equation (18). Then (i)–(iv) of Proposition 6.1 hold if we replace $\Phi^{-1}(C_\theta)$ by $\Phi^{-1}(C_\theta) - \Gamma - \overline{\Gamma}$, where we set $\Gamma := \Phi^{-1}(P_\infty)$ and $\overline{\Gamma} := \Phi^{-1}(\overline{P}_\infty)$.*

Proof. (i) can be proved in the same way as in Proposition 6.1. (But in this case, any small resolution $Z \to Z_0$ gives the resolution of $\Phi_0^{-1}(C_0)$ at the points over P_∞ and $\overline{P}_\infty$, as will be mentioned below.) For (ii) first note that we have $\Phi^{-1}(C_\theta) = L_1 + L_2 + \Gamma + \overline{\Gamma} \in |-2K|$ this time, where L_1 and L_2 are irreducible components of $\Phi^{-1}(C_\theta) - \Gamma - \overline{\Gamma}$. Γ and $\overline{\Gamma}$ are chains of three (-2)-curves on $S_\lambda = \Phi^{-1}(H_\lambda)$, since they are exceptional curves of the minimal resolution of A_3-singularities of surface. We write $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where Γ_i's are smooth rational curves satisfying $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_3 = 1$ and $\Gamma_1\Gamma_3 = 0$ on S_λ. We then have $\Gamma^2 = \overline{\Gamma}^2 = -2$. Furthermore, as we will state and prove in Lemma 7.14, we have (or more precisely, can suppose) $L_1\Gamma_1 = 1$, $L_1\Gamma_i = 0$ for $i = 2, 3$, and $L_3\Gamma_3 = 1$, $L_3\Gamma_i = 0$ for $i = 1, 2$. (In that lemma we write L_θ^+ and L_θ^- instead of L_1 and L_2 respectively.) It follows that $L_1\Gamma = L_1\overline{\Gamma} = L_2\Gamma = L_2\overline{\Gamma} = 1$. Therefore again by Proposition 3.5, we get $8 = (-2K)^2 = (L_1 + L_2 + \Gamma + \overline{\Gamma})^2 = L_1^2 + L_2^2 + 2L_1L_2 + 4$. But because L_1 and L_2 intersect transversally at two points (over the touching points of C_θ and B_λ), and because L_1 and L_2 do not intersect on $\Gamma \cup \overline{\Gamma}$, we have $L_1L_2 = 2$. Therefore we have $L_1^2 + L_2^2 = 0$. Hence by the same reason in the proof of the previous proposition, we again have $L_1^2 = L_2^2 = 0$. This implies (ii). (iii) follows from (ii), because we have $L_1L_2 = 2$ as is already seen. (iv) can be proved by the same idea as in the previous proposition: first we may assume $\theta = 0$. Then

by (18) the equation of C_0 on $V = \{y_1 \neq 0\} = \{(v_2, v_3)\}$ is given by

$$\sqrt{Q^2 - f} + \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}} \cdot (v_2 + v_3) + v_2 v_3 = 0.$$

If we use another coordinate (u, v) defined in the proof of the previous proposition, this can be written as

$$C_0: \ v^2 = u^2 + 2\sqrt{2}\sqrt{\sqrt{Q^2 - f} - Q} \cdot u + 4\sqrt{Q^2 - f}. \tag{30}$$

Next we introduce a new variable w by setting $u = -\sqrt{2}(\sqrt{Q^2 - f} - Q)^{\frac{1}{2}} + iw$. Then the equation becomes

$$C_0: \quad v^2 + w^2 = 2\left(\sqrt{Q^2 - f} + Q\right). \tag{31}$$

Put $\mathcal{C} := C_0 \cap \mathbf{R}^2$, where $\mathbf{R}^2 = \{(w, v) \,|\, w \in \mathbf{R}, v \in \mathbf{R}\}$. Then since $\sqrt{Q^2 - f} + Q > 0$, $\mathcal{C}$ is a real circle in $\{(w, v) \in \mathbf{R}^2\}$.

By (26) we have

$$\Phi_0^{-1}(V): \ z^2 = f - \left(\frac{u^2 - v^2}{4} + Q\right)^2. \tag{32}$$

On the other hand, by (30), we have

$$C_0: \ u^2 - v^2 = -2\sqrt{2}\sqrt{\sqrt{Q^2 - f} - Q} \cdot u - 4\sqrt{Q^2 - f}$$

on C_0. Substituting this into (32), we get

$$\Phi_0^{-1}(C_0): \ z^2 = f + \frac{\sqrt{Q^2 - f} - Q}{2} w^2.$$

Then by using (31), we get

$$\Phi_0^{-1}(C_0): \ z^2 = -\frac{\sqrt{Q^2 - f} - Q}{2} v^2. \tag{33}$$

Therefore, z is pure imaginary over $\mathcal{C}$, so that we can distinguish two sheets by looking the sign of z/i. By (33) and (31), the branch points of $\Phi^{-1}(C_0) \to C_0$ is the two points $(w, v) = (\pm(2(\sqrt{Q^2 - f} + Q))^{\frac{1}{2}}, 0)$ which lie on $\mathcal{C}$. The real structure is given by $(w, v) \mapsto (-\overline{w}, -\overline{v})$ and this is equal to $(-w, -v)$ on $\mathcal{C}$. Thus the real structure on $\mathcal{C}$ interchanges the two segments separated by the two branch points. Moreover, the real structure on the fiber coordinate is given by $z \mapsto \overline{z}$. Therefore, it changes the sign of z/i over $\mathcal{C}$. This implies that each component of $\Phi^{-1}(\mathcal{C})$ is real. Therefore that of $\Phi^{-1}(C_0)$ is also real. $\square$

The situation slightly changes for touching conics of orbit type:

Proposition 6.3. *Suppose $f(\lambda) \neq 0$ and let $C_\alpha \subset H_\lambda$ be a real touching conic of orbit type given by the equation (25). Then we have: (i) C_α is contained in B iff $\alpha = -Q \pm \sqrt{f}$. In the following (ii)–(vi) suppose that $\alpha \neq -Q \pm \sqrt{f}$. (ii) $\Phi^{-1}(C_\alpha) - \Gamma - \overline{\Gamma}$ has just two irreducible components, both of which are smooth rational curves that are mapped biholomorphically onto C_α. (Here Γ and $\overline{\Gamma}$ are as in Proposition 6.2.) (iii) Each irreducible component of $\Phi^{-1}(C_\alpha) - \Gamma - \overline{\Gamma}$ has a trivial normal bundle in $S_\lambda = \Phi^{-1}(H_\lambda)$. (iv) The two irreducible components of $\Phi^{-1}(C_\alpha) - \Gamma - \overline{\Gamma}$ belong to one and the same pencil on S_λ. (v) Each irreducible component of $\Phi^{-1}(C_\alpha) - \Gamma - \overline{\Gamma}$ is real iff $f > 0$ and $-Q - \sqrt{f} < \alpha < -Q + \sqrt{f}$ are satisfied. (vi) There is no real point on C_α if f and α satisfies the inequalities of (v). (vii) If $f > 0$ and $\alpha = -Q \pm \sqrt{f}$, then C_α has no real point.*

Note that $f > 0$ implies $Q \geq \sqrt{f}$ by Proposition 2.5 (ii).

Proof. (i) By substituting $y_2y_3 = \alpha y_1^2$ into the defining equation of B_λ, we obtain $\left((\alpha+Q)^2 - f\right) y_1^4 = 0$. Thus if C_α is contained in B iff $(\alpha + Q)^2 = f$, which implies $\alpha = -Q \pm \sqrt{f}$, as desired. (ii) can be seen in the same way as in (i) of Proposition 6.1. (This time, any small resolution $Z \to Z_0$ gives the normalization of $\Phi_0^{-1}(C_0)$.) Next we prove (iii). Let $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, $\overline{\Gamma} = \overline{\Gamma}_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$, L_1 and L_2 have the same meaning as in the proof of the last proposition. It is obvious that Γ and $\overline{\Gamma}$ are disjoint. In Lemma 7.19 we will show that L_1 is disjoint from Γ_1 and Γ_3, and the same for L_2, and both of L_1 and L_2 intersect transversally at a unique point on Γ_2. (In the lemma we write L_α^+ and L_α^- instead of L_1 and L_2.) By reality, the same is true for the intersection of L_1 and L_2 with $\overline{\Gamma}$. Moreover, it will also be shown in Lemma 7.19 that if L_1 and L_2 are different, these two curves are disjoint. Thus we have $L_1L_2 = \Gamma\overline{\Gamma} = L_1\Gamma_1 = L_1\overline{\Gamma}_1 = L_2\Gamma_1 = L_2\overline{\Gamma}_1 = L_1\Gamma_3 = L_1\overline{\Gamma}_3 = L_2\Gamma_3 = L_2\overline{\Gamma}_3 = 0$ on S_λ, while $L_1\Gamma_2 = L_1\overline{\Gamma}_2 = L_2\Gamma_2 = L_2\overline{\Gamma}_2 = 1$ (also on S_λ). In particular, we have $L_1\Gamma = L_1\overline{\Gamma} = L_2\Gamma = L_2\overline{\Gamma} = 1$. On the other hand, we still have $8 = (-2K)^2 = (L_1 + L_2 + \Gamma + \overline{\Gamma})^2$ and $L_1^2 \geq 0$ and $L_2^2 \geq 0$. Combining these, we get $L_1^2 = L_2^2 = 0$ on S_λ. Thus we have (iii). (iv) easily follows if we consider the linear systems $|L_1|$ and $|L_2|$, and if we note that $L_1L_2 = 0$. Next we show (v). Substituting $v_2v_3 = \alpha$ into (26), we get $z^2 + (\alpha+Q)^2 - f = 0$. From this, the equations of irreducible components of $\Phi_0^{-1}(C_\alpha)$ can be calculated to be

$$z = \pm\sqrt{f - (\alpha+Q)^2} \tag{34}$$

Recalling that the real structure is given by $z \mapsto \overline{z}$, these curves are real iff $f - (\alpha+Q)^2 \geq 0$. In particular, $f \geq 0$ follows. Then we have $-Q - \sqrt{f} < \alpha < -Q + \sqrt{f}$ under our assumption, and we get (v) and (vii). Finally, (vi) immediately follows from Lemma 2.4. □

Proposition 6.3 implies that not all touching conics of orbit type can be the image of a twistor line: $f(\lambda) > 0$ is needed, and further, $-Q - \sqrt{f} \leq \alpha \leq -Q + \sqrt{f}$ must be satisfied. But once we know that one of the two irreducible components is actually a twistor line, it follows that the other component is also a twistor line, since by (iv) these two components can be connected by deformation in S_λ (hence also in Z) preserving the real structures. This is not true for touching conics of generic type and special type, because the two irreducible components of $\Phi^{-1}(C_\theta)$ or $\Phi^{-1}(C_\theta) - \Gamma - \overline{\Gamma}$ intersect.

7. Twistor lines lying on $\mathbf{C}^*$-invariant fundamental divisors

In this section we calculate the normal bundle of L in Z, where L is a real irreducible component of the inverse images of the real touching conics which are determined in Section 5. We continue to use the notations in the previous section. Because touching conics in $H_\lambda \in \langle l_\infty \rangle^\sigma$ generally go through the singular points (P_∞ and $\overline{P}_\infty$) of B, the normal bundles generally depend on the choice of a small resolution of Z_0. So in this section (especially in Section 7.3–7.6), we need to discuss the choice of small resolution of the conjugate pair of singularities of Z_0. (But we do not discuss resolution of the unique real ordinary double point of Z_0. This is discussed in the next section.)

Let us briefly describe the content of this section. In Section 7.1 we give a simple lemma which will be used to determine the normal bundles. Some notations are also introduced. In Sections 7.2–7.4 we determine the normal bundles, according to the three types of real touching conics. These subsections are organized as follows: first we explicitly calculate the intersection of irreducible components (of the inverse images of touching conics) with some curves. Consequently we get a function of λ (which will be written h_i). Second we show that the normal bundle in problem degenerates into $O \oplus O(2)$ precisely when λ is a critical point of this function. Then we determine the critical points. In Section 7.5 we use the results in Sections 7.2–7.4 to determine which type of touching conics actually come from twistor lines (Proposition 7.23). We also prove that among various ways there are only two

small resolutions of the conjugate pair of singularities of Z_0 which can yield twistor space (Proposition 7.24). Then finally in Section 7.6 we show that our candidates of twistor lines obtained in Section 7.5 actually form a connected family (Proposition 7.30); namely they can be connected by deformations in the threefold, by adding the inverse image of four tropes of B. This result is used in Section 9 to construct *arbitrary* twistor lines in Z.

7.1. **Preliminary lemma and notations.** In order to determine which one of $O(1)^{\oplus 2}$ and $O \oplus O(2)$ actually occurs, we use the following elementary criterion:

Lemma 7.1. *Let $E \to \mathbf{CP}^1$ be a holomorphic line bundle of rank two, and assume that E is isomorphic to $O(1)^{\oplus 2}$ or $O \oplus O(2)$. Let s and t be global sections of E which are linearly independent as global sections. Then $E \simeq O \oplus O(2)$ iff there are constants $a, b \in \mathbf{C}$ such that $as + bt$ has two zeros.*

Proof. It is immediate to see that non-zero sections of $O(1)^{\oplus 2}$ have at most one zero. So sufficiency follows. Conversely let $s = (s_1, s_2)$ and $t = (t_1, t_2)$ be any linearly independent sections of $O \oplus O(2)$, where $s_1, t_1 \in \Gamma(O) = \mathbf{C}$ and $s_2, t_2 \in \Gamma(O(2))$. Then take $a, b \in \mathbf{C}$ such that $as_1 + bt_1 = 0$. Then $as + bt$ can be regarded as a non-zero section of $O(2)$ so that it has two zeros. □

Next we introduce some notations which will be used throughout this section. As in the previous sections, $\lambda \in \mathbf{R}$ denotes a parameter on the space of real $U(1)$-invariant planes. In other words, λ is a parameter on the real locus l_0^σ of the real line $l_0 := \{y_2 = y_3 = 0\}$. The function $f(\lambda) = \lambda(\lambda+1)(\lambda - a)$, $(a > 0)$ defines four open intervals in the circle l_0^σ:

$$I_1 = (-\infty, -1),\ I_2 = (-1, 0),\ I_3 = (0, a) \ \text{ and } \ I_4 = (a, +\infty).$$

Namely, $I_1 \cup I_3 = \{\lambda \in \mathbf{R} \,|\, f(\lambda) < 0\}$ and $I_2 \cup I_4 = \{\lambda \in \mathbf{R} \,|\, f(\lambda) > 0\}$. By Proposition 2.3, the equation $Q(\lambda, 1)^2 - f(\lambda) = 0$ has a unique real solution $\lambda = \lambda_0$ which is necessarily a double root. Since we have $f(\lambda_0) = Q(\lambda_0, 1)^2 > 0$, $\lambda_0 \in I_2 \cup I_4$. By applying a projective transformation $(y_0, y_1) \mapsto (ay_1, -y_0)$ (interchanging I_2 and I_4) if necessary, we may suppose that $\lambda_0 \in I_4$. Then we set $I_4^- = (a, \lambda_0)$ and $I_4^+ = (\lambda_0, +\infty)$.

Next suppose $\lambda \in I_2 \cup I_4^- \cup I_4^+$, and let

$$\mathcal{C}_\lambda^{\mathrm{gen}} = \{C_\theta \subset H_\lambda \,|\, C_\theta \text{ is defined by (10)}\}$$

be the set of real touching conics of generic type on H_λ. Note that if $\lambda = \lambda_0$ or if $\lambda \in I_1 \cup I_3$ there is no real touching conic of generic type on H_λ by Proposition 5.2 (ii). Similarly, for $\lambda \in I_1 \cup I_3$, let

$$\mathcal{C}_\lambda^{\mathrm{sp}} = \{C_\theta \subset H_\lambda \,|\, C_\theta \text{ is defined by (18)}\}$$

be the set of real touching conics of special type on H_λ. Note that if $\lambda \in I_2 \cup I_4$ there is no real touching conic of generic type by Proposition 5.4 (i). Finally for $\lambda \in I_2 \cup I_4$, let

$$\mathcal{C}_\lambda^{\mathrm{orb}} = \left\{ C_\alpha \subset H_\lambda \,|\, -Q - \sqrt{f} \le \alpha \le -Q + \sqrt{f},\ C_\theta \text{ is defined by (25)} \right\}$$

be the set of real touching conics of orbit type on H_λ. Note that the restriction on α implies that the two irreducible components of the inverse image are respectively real (Proposition 6.3 (v)). $\mathcal{C}_\lambda^{\mathrm{gen}}$ and $\mathcal{C}_\lambda^{\mathrm{sp}}$ are parametrized by a circle on which $U(1)$ naturally acts transitively, whereas $\mathcal{C}_\lambda^{\mathrm{orb}}$ is parametrized by a closed interval on which $U(1)$ acts trivially.

7.2. **The case of generic type.** This is the easiest case since touching conics of generic type do not pass through the singular points of B_0 and hence resolutions of Z_0 do not have effect on the normal bundles. Suppose $\lambda \in I_2 \cup I_4^- \cup I_4^+$ and take $C_\theta \in \mathcal{C}_\lambda^{\mathrm{gen}}$. First we calculate the intersection of C_θ and l_∞, where l_∞ is the real line defined by $y_0 = y_1 = 0$ as before. Let $x_2 = y_2/y_3$ be a non-homogeneous coordinate on l_∞ (around $P_\infty = (0:0:0:1)$).

Lemma 7.2. *The set $\{C_\theta \cap l_\infty \,|\, C_\theta \in \mathcal{C}_\lambda^{\rm gen}\}$ consists of disjoint two circles about P_∞ in l_∞, whose radiuses (with respect to the coordinate x_2 above) are given by*

$$h_0(\lambda) := \frac{Q + \sqrt{Q^2 - f}}{\sqrt{f}} \quad \text{and} \quad h_0(\lambda)^{-1} = \frac{Q - \sqrt{Q^2 - f}}{\sqrt{f}}$$

respectively, where we put $Q = Q(\lambda, 1)$ and $f = f(\lambda)$ as before.

Note that we have $Q^2 - f > 0$ and $Q > \sqrt{f}$ by Proposition 2.6, and therefore $h_0 > 1 > h_0^{-1} > 0$ holds. Moreover, h_0 and h_0^{-1} are differentiable on $I_2 \cup I_4^- \cup I_4^+$.

Proof. On $H_\lambda = \{(y_0 : y_1 : y_2)\}$, l_∞ is defined by $y_1 = 0$. Therefore by (10) we readily have

$$C_\theta \cap l_\infty = \left\{ x_2 = \frac{-Q \pm \sqrt{Q^2 - f}}{\sqrt{f}} \cdot e^{-i\theta} \right\}. \tag{35}$$

This directly implies the claim of the lemma. □

By Proposition 6.1, $\Phi^{-1}(C_\theta)$ consists of two irreducible components, both of which are real rational curves. We denote these components by L_θ^+ and L_θ^-, although there is no canonical way of making a distinction of these two. Again by Proposition 6.1, L_θ^+ and L_θ^- respectively form disjoint families

$$\mathcal{L}_\lambda^+ = \{L_\theta^+ \,|\, \theta \in \mathbf{R}\} \quad \text{and} \quad \mathcal{L}_\lambda^- = \{L_\theta^- \,|\, \theta \in \mathbf{R}\}$$

of (real and smooth) rational curves on $S_\lambda = \Phi^{-1}(H_\lambda)$. These are real members of real pencils on S_λ and each member has no real point by Proposition 5.2 (iii). Because $U(1)$ acts also on the parameter spaces ($= S^1$) of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$, the normal bundles of L_θ^+ and L_θ^- inside Z_0 are independent of the choice of θ. The following proposition plays a key role in determining the normal bundle:

Proposition 7.3. *For any $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$, the normal bundle of L in Z_0 is isomorphic to either $O(1)^{\oplus 2}$ or $O \oplus O(2)$. Further, the latter holds iff λ is a critical point of $h_0(\lambda)$ defined in Lemma 7.2.*

In particular, members of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ have the same normal bundle in Z_0.

Proof. L is contained in the smooth surface $S_\lambda = \Phi^{-1}(H_\lambda)$ and therefore we have an exact sequence $0 \to N_{L/S_\lambda} \to N_{L/Z_0} \to N_{S_\lambda/Z_0}|_L \to 0$. By Proposition 6.1 (ii), we have $N_{L/S_\lambda} \simeq O_L$. On the other hand, S_λ is a smooth member of $|(-1/2)K_Z|$. Therefore by adjunction formula we have $K_{S_\lambda} \simeq K_Z|_{S_\lambda} \otimes N_{S_\lambda/Z} \simeq K_Z|_{S_\lambda} \otimes (-1/2)K_Z|_{S_\lambda}$ and hence $N_{S_\lambda/Z} \simeq (-1/2)K_Z|_{S_\lambda} \simeq -K_{S_\lambda}$. Hence we get $N_{S_\lambda/Z}|_L \simeq -K_{S_\lambda}|_L \simeq -K_L \otimes N_{L/S} \simeq O_L(2)$. Therefore by the short exact sequence above, $N_\lambda := N_{L/Z}$ is isomorphic to either $O \oplus O(2)$ or $O(1)^{\oplus 2}$. Thus we get the first claim of the proposition.

In order to show the second claim, we recall three facts; the first one is about the the natural real structure on $\Gamma(N_\lambda)$, the space of sections of N_λ. Since L is real, σ naturally acts on $\Gamma(N_\lambda)$ as the complex conjugation. For $s \in \Gamma(N_\lambda)$ we denote by ${\rm Re} s$ and ${\rm Im} s$ the real part and the imaginary part of s respectively. Namely, ${\rm Re} s = (s + \overline{\sigma(s)})/2$ and ${\rm Im} s = (s - \overline{\sigma(s)})/2$. Secondly, recall that $L \cap \Phi^{-1}(l_\infty)$ consists of a conjugate pair of points, one of which corresponds to the point of l_θ satisfying $x_2 = h_2(\lambda)e^{i\theta}$, and the other one corresponds to to the point of l_θ satisfying $x_2 = h_2^{-1}(\lambda)e^{i\theta}$. (See (35)). Let z_λ and $\overline{z}_\lambda$ be the former and the latter point respectively. Thirdly recall that any one-parameter family of holomorphic deformation of L in Z naturally gives rise to a holomorphic section of N_λ: roughly this section is obtained by taking the tangent vector of the 'orbit' for each point of L. More precisely, take a neighborhood U of each point of L and a holomorphic coordinate (z_1, z_2, z_3) on U, such that each L_x, $|x| < \epsilon$ with $L_0 = L$ of the given one-parameter family

is defined by $z_2 = f(z_1, x)$ and $z_3 = g(z_1, x)$ satisfying $f(z_1, 0) \equiv g(z_1, 0) \equiv 0$. Then a representative of the section of N_λ is given by $(\partial f/\partial x, \partial g/\partial y)$.

Now we have the following two one-parameter families of deformations of L in Z: the first one is obtained by moving L by $\mathbf{C}^*$-action, where the $\mathbf{C}^*$-action is the complexification of the $U(1)$-action. The second one is obtained by moving the parameter λ in $\mathbf{C}$, while fixing θ. Let $s \in \Gamma(N_\lambda)$ and $t \in \Gamma(N_\lambda)$ be the holomorphic sections associated to the former and the latter family respectively. These are linearly independent sections, since each representative of s is tangent to S_λ (by the $\mathbf{C}^*$-invariance of S_λ), whereas that of t is not. Because the $\mathbf{C}^*$-action preserves S_λ, it follows from Proposition 6.1 (ii) and (iii) that each of the curves of the former family are disjoint. This implies that s is nowhere vanishing. Next we consider the latter family. First, noting $H_\lambda \cap H_{\lambda'} = l_\infty$ for $\lambda \neq \lambda'$, t can be zero only on $\Phi^{-1}(l_\infty)$ (cf. Figure 1 in Section 2). Suppose λ is a critical point of h_0; namely $h_0'(\lambda) = 0$, where the derivative is with respect to real λ, of course. Then, since h is a holomorphic function of λ, it can be easily derived from the Cauchy-Riemann equation that $(\partial h_0/\partial\lambda)(\lambda) = 0$. By the way how we take a representative of the section of N explained in the previous paragraph, this directly implies that t vanishes at z_λ. On the other hand, $h_0'(\lambda) = 0$ implies $(h_0^{-1})'(\lambda) = 0$. Then in the same manner as above, we have $(\partial h_0^{-1}/\partial\lambda)(\lambda) = 0$. These imply that t also vanishes at $\overline{z}_\lambda$ and we have obtained $t(z_\lambda) = t(\overline{z}_\lambda) = 0$. Therefore by Lemma 7.1, we get $N_\lambda \simeq O \oplus O(2)$.

Next suppose $h_0'(\lambda) \neq 0$, so that $(h_0^{-1})'(\lambda) \neq 0$. We claim that $\mathrm{Re}(bs + t)$ cannot vanish at z_λ and $\overline{z}_\lambda$ simultaneously, for any $b \in \mathbf{C}$. Because C_θ intersects l_∞ transversally, t also becomes a nowhere vanishing section under our assumption. Hence the claim is true for $b = 0$. Putting $b = b_1 + ib_2$, $b_1, b_2 \in \mathbf{R}$, we easily get

$$\mathrm{Re}(bs + t) = b_1 \mathrm{Re}s + (\mathrm{Re}t - b_2 \mathrm{Im}s). \tag{36}$$

Since s comes from the $\mathbf{C}^*$-action, and since its real part corresponds to the $U(1)$-action, $(\mathrm{Re}s)(z_\lambda)$ is represented by the tangent vector of the $U(1)$-orbit going through z_λ. On the other hand, (35) implies that $(\mathrm{Re}t)(z_\lambda)$ is represented by a tangent vector which is parallel to $(\mathrm{Im}s)(z_\lambda)$. Hence by (36), we can deduce that $\mathrm{Re}(bs + t)(z_\lambda) = 0$ implies $b_1 = 0$ and

$$(\mathrm{Re}t)(z_\lambda) = b_2(\mathrm{Im}s)(z_\lambda). \tag{37}$$

Similarly, $\mathrm{Re}(bs + t)(\overline{z}_\lambda) = 0$ implies $b_1 = 0$ and

$$(\mathrm{Re}t)(\overline{z}_\lambda) = b_2(\mathrm{Im}s)(\overline{z}_\lambda). \tag{38}$$

Suppose $b_2 > 0$. Then since $\{\mathrm{Re}s(z_\lambda), \mathrm{Im}s(z_\lambda)\}$ is an oriented basis of $T_{z_\lambda}(\Phi^{-1}(l_\infty))$ from the beginning, (37) implies that $\{\mathrm{Re}s(z_\lambda), \mathrm{Re}t(z_\lambda)\}$ is an oriented basis of $T_{z_\lambda}(\Phi^{-1}(l_\infty))$. Further, we have

$$(\mathrm{Re}s)(\overline{z}_\lambda) = \sigma_*((\mathrm{Re}s)(z_\lambda)) \quad \text{and} \quad (\mathrm{Re}t)(\overline{z}_\lambda) = \sigma_*((\mathrm{Re}t)(z_\lambda)).$$

Hence we get by (38)

$$(\mathrm{Im}s)(\overline{z}_\lambda) = \frac{1}{b_2}(\mathrm{Re}t)(\overline{z}_\lambda) = \frac{1}{b_2}\sigma_*\left((\mathrm{Re}t)(z_\lambda)\right).$$

So we have

$$\{(\mathrm{Re}s)(\overline{z}_\lambda), (\mathrm{Im}s)(\overline{z}_\lambda)\} = \{\sigma_*((\mathrm{Re}s)(z_\lambda)), \sigma_*((\mathrm{Re}t)(z_\lambda))/b_2\}.$$

But since σ is anti-holomorphic, σ is orientation reversing. Further, as is already seen, $\{\mathrm{Re}s(z_\lambda), \mathrm{Re}t(z_\lambda)\}$ is an oriented basis of $T_{z_\lambda}(\Phi^{-1}(l_\infty))$ (if $b_2 > 0$). This implies that $\{\sigma_*((\mathrm{Re}s)(z_\lambda)), \sigma_*((\mathrm{Re}t)(z_\lambda))/b_2\}$ is an anti-oriented basis of $T_{\overline{z}_\lambda}(\Phi^{-1}(l_\infty))$. This contradicts to the fact that $\{(\mathrm{Re}s)(\overline{z}_\lambda), (\mathrm{Im}s)(\overline{z}_\lambda)\}$ is an oriented basis of $T_{\overline{z}_\lambda}(\Phi^{-1}(l_\infty))$. Therefore, $\mathrm{Re}(bs + t)$ cannot vanish at z_λ and $\overline{z}_\lambda$ simultaneously, provided $b_2 > 0$. Parallel arguments show the same claim holds for the case $b_2 < 0$. Thus we have shown that $\mathrm{Re}(bs + t)$ cannot vanish at z_λ and $\overline{z}_\lambda$ at the same time, as claimed. On the other hand, it is obvious that

$bs+t$ does not vanish except $\{z_\lambda, \overline{z}_\lambda\}$. Therefore, the zero locus of $bs+t$ consists of at most one point for any $b \in \mathbf{C}$. Since s is a nowhere vanishing section, Lemma 7.1 implies $N_\lambda \simeq O(1)^{\oplus 2}$. □

Lemma 7.4. *Let $h_0 = h_0(\lambda)$ be the positive valued function on $I_2 \cup I_4$ defined in Lemma 7.2, which is differentiable on $I_2 \cup I_4 \backslash \{\lambda_0\}$. Then h_0 has a unique critical point on I_2, and has no critical point on $I_4 \backslash \{\lambda_0\}$. (See Figure 7.)*

Proof. Although elementary, we include the proof since it is not so easy (at least for the author) and it needs the previous result (Proposition 2.5). We have $Q(-1) > 0$, $Q(0) > 0$ and $Q(a) > 0$ by Proposition 2.5 (i), and

$$h_0 = \frac{Q + \sqrt{Q^2 - f}}{\sqrt{f}} = \frac{Q}{\sqrt{f}} + \sqrt{\frac{Q^2}{f} - 1}. \tag{39}$$

From these, it follows that $\lim_{\lambda \downarrow -1} h_0(\lambda) = \lim_{\lambda \uparrow 0} h_0(\lambda) = +\infty$. Therefore h_0 has at least one critical point on I_2, since h_0 is differentiable on I_2. So to prove the lemma it suffices to show that this is a unique critical point on $I_2 \cup I_4 \backslash \{\lambda_0\}$.

We consider the real valued function $\gamma := Q^2/f$ defined on $I := I_1 \cup I_2 \cup I_3 \cup I_4$, which is clearly differentiable on I. Then $h_0 = \sqrt{\gamma} + \sqrt{\gamma - 1}$ on $I_2 \cup I_4$, and we have

$$h_0' = \gamma' \cdot \left(\frac{1}{2\sqrt{\gamma}} + \frac{1}{2\sqrt{\gamma - 1}} \right),$$

provided $\lambda \neq \lambda_0$. Therefore on $I_2 \cup I_4 \backslash \{\lambda_0\}$, $h_0'(\lambda) = 0$ iff $\gamma'(\lambda) = 0$. It is readily seen that $\lim_{\lambda \downarrow -\infty} \gamma(\lambda) = \lim_{\lambda \uparrow -1} \gamma(\lambda) = -\infty$, $\lim_{\lambda \downarrow -1} \gamma(\lambda) = \lim_{\lambda \uparrow 0} \gamma(\lambda) = \infty$, $\lim_{\lambda \downarrow 0} \gamma(\lambda) = \lim_{\lambda \uparrow (a)} \gamma(\lambda) = -\infty$, and $\lim_{\lambda \downarrow (a)} \gamma(\lambda) = \lim_{\lambda \uparrow \infty} \gamma(\lambda) = \infty$. Therefore γ has at least one critical point on each I_j, $1 \leq j \leq 4$. We also have

$$\gamma' = Q(2Q'f - Qf')/f^2. \tag{40}$$

Suppose that critical points of γ on I_2 are not unique. Then γ has at least three critical points on I_2. This implies that γ has at least four critical points on $I_2 \cup I_4$. Because $Q > 0$ on $I_2 \cup I_4$ (Proposition 2.5 (i)), these critical points must correspond to zeros of $2Q'f - Qf'$ whose degree is just four. By (40) this implies that the other critical points of γ on I_1 and I_3 must correspond to zeros of Q. But this cannot happen since $Q > 0$ on $I_2 \cup I_4$ and since Q is degree two. Therefore, our assumption fails and it follows that critical points on γ on I_2 is unique. Hence critical points of h_0 on I_2 is also unique. Exactly the same argument shows that γ has a unique critical point on I_4. This critical point must be λ_0, since γ attains the minimal value $(= 1)$ there. This implies that g has no critical point on $I_4 \backslash \{\lambda_0\}$. Thus we obtain the claims of the lemma. □

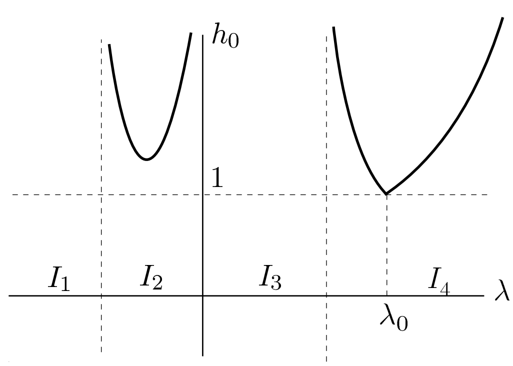

FIGURE 7. behavior of h_0

We use the lemma to prove the main result of this subsection:

Proposition 7.5. *(i) If $\lambda \in I_4$ and if $\lambda \neq \lambda_0$, we have $N_{L/Z} \simeq O(1)^{\oplus 2}$ for any $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$. (ii) There is a unique $\lambda \in I_2$ such that $N_{L/Z} \simeq O \oplus O(2)$ for any $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$. For any other $\lambda \in I_2$, we have $N_{L/Z} \simeq O(1)^{\oplus 2}$ for arbitrary $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$. (iii) If $\lambda \in I_2$, any member of $\mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$ is not a twistor line in Z (even if Z is actually a twistor space).*

Proof. (i) and (ii) are direct consequences of Proposition 7.3 and Lemma 7.4. To show (iii), let $\lambda' \in I_2$ be the unique critical point of g. Then by (ii), any $L' \in \mathcal{L}_{\lambda'}^+ \cup \mathcal{L}_{\lambda'}^-$ is not a twistor line. We can see that for any $\lambda \in I_2$ and for any $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$, L can be deformed into some $L' \in \mathcal{L}_{\lambda'}^+ \cup \mathcal{L}_{\lambda'}^-$ preserving the real structure. In fact, we have $\Phi(L) = C_\theta$ for some $C_\theta \in \mathcal{C}_\lambda^{\text{gen}}$. Then since I_2 is an interval in $\mathbf{R}$, C_θ can be canonically deformed into some $C'_\theta \in \mathcal{C}_{\lambda'}^{\text{gen}}$. (The point is that we take a constant θ for any $\lambda \in I_2$.) Correspondingly, we obtain deformation of L into $L' \in \mathcal{L}_{\lambda'}^+ \cup \mathcal{L}_{\lambda'}^-$ such that $\Phi(L') = C'_\theta$. Thus we get an explicit real one-dimensional family of rational curves in Z containing L and L' as its members, as claimed. Any member of this family is real by Proposition 6.1 (iv). Since any deformation of twistor line preserving the real structure is still a twistor line, it follows that L is not a twistor line. $\square$

We note the proof of (iii) does not work for I_4, since as λ goes to λ_0, the curve C_θ (defined by (10)) degenerates into a double line. This is an important point for our global construction of arbitrary twistor lines in Z.

Corollary 7.6. *If $\lambda \in I_2$, only the members of $\mathcal{C}_\lambda^{\text{orb}}$ can be the image of twistor lines. Namely over I_2, members of $\mathcal{C}_\lambda^{\text{gen}}$ cannot be the images of twistor lines.*

Proof. By Proposition 5.6 (i), the image of a twistor line contained in H_λ is either a touching conic of generic type or that of orbit type for $\lambda \in I_2 \cup I_4$. But by Proposition 7.5 (iii) the former cannot be the image of a twistor line if $\lambda \in I_2$. $\square$

Corollary 7.7. *A twistor line of a self-dual 4-manifold (i.e. a fiber of the twistor fibration) is not in general characterized by the property that it is a real smooth rational curve without real point whose normal bundle is isomorphic to $O(1)^{\oplus 2}$. More concretely, the twistor space of any non-LeBrun self-dual metric on $3\mathbf{CP}^2$ of positive scalar curvature with a non-trivial Killing field always possesses such a real rational curve.*

Proof. Let Z be a twistor space as in the corollary. Then Z has a structure as in Proposition 2.1, where Q and a satisfy the conditions in Proposition 2.6. By (ii) and (iii) of Proposition 7.5, Z always has a real smooth rational curve L satisfying $N_{L/Z} \simeq O(1)^{\oplus 2}$, but which is not a twistor line. This L has no real point by Proposition 5.2 (iii) and the reality of Φ. $\square$

Next we give another geometric proof for the fact that L cannot be a twistor line for $\lambda \in I_2$ (although we will not need this result in the sequel).

Proposition 7.8. *If $\lambda \in I_2$ is not a critical point of h_0, there exists a unique $\mu \in I_2$ with $\lambda \neq \mu$ satisfying the following: for any $L \in \mathcal{L}_\lambda^+$ (resp. $L \in \mathcal{L}_\lambda^-$) there exists $L' \in \mathcal{L}_\mu^+$ (resp. $L' \in \mathcal{L}_\mu^-$) such that $L \cap L' \neq \phi$.*

Proof. Let $\lambda' \in I_2$ be the unique critical point of h_0 as before. By our proof of Lemma 7.4 we have $\lim_{\lambda \downarrow -1} h_0(\lambda) = \lim_{\lambda \uparrow 0} h_0(\lambda) = +\infty$ and g is strictly decreasing on $(-1, \lambda')$ and strictly increasing on $(\lambda', 0)$. Suppose $\lambda < \lambda'$. Let Ξ and $\overline{\Xi}$ be the conjugate pair of rational curves which are mapped biholomorphically onto l_∞. (See Proposition 3.2.) Then by Lemma 7.2, $L \cap \Xi$ is a point which is either $x_2 = h_0(\lambda)e^{i\theta}$ or $x_2 = h_0(\lambda)^{-1}e^{i\theta}$ for some $\theta \in \mathbf{R}$, where we identify Ξ and l_∞ via Φ and use $x_2 = y_2/y_3$ as an affine coordinate on l_∞ as before. If $x_2 = h_0(\lambda)^{-1}e^{i\theta}$, $\overline{\Xi} \cap L$ is a point having $x_2 = h_0(\lambda)e^{i\theta}$. Thus (by a possible exchange of Ξ and $\overline{\Xi}$) we may suppose that $\Xi \cap L$ is a point satisfying $x_2 = h_0(\lambda)e^{i\theta}$. Then by the behavior of h_0 mentioned above, there exists a unique $\mu > \lambda'$, $\mu \in I_2$ such that $h_0(\lambda) = h_0(\mu)$.

On the other hand, by our choice of L we have $\Phi(L) = C_\theta$ for some $C_\theta \in \mathcal{C}_\lambda^{\rm orb}$. Then take $L' \in \mathcal{L}_\mu^+$ such that $\Phi(L') = C_\theta \in \mathcal{C}_\mu^{\rm orb}$. (Although we use the same symbol C_θ, they represent different conics since $\lambda \neq \mu$. The point is that we take the same θ for different λ's.) Then $L \cap L' \cap \Xi$ is a point satisfying $x_2 = h_0(\lambda) e^{i\theta}$. (We also have $L \cap L' \cap \overline{\Xi}$ is a point satisfying $x_2 = h_0(\lambda)^{-1} e^{i\theta}$.) Thus we have proved the claim for $\lambda < \lambda'$. Of course, the case $\lambda > \lambda'$ and the case $L \in \mathcal{L}_\lambda^-$ are similar. □

The proposition shows that when $\lambda \in I_2$ passes through the critical point ($= \lambda'$) of h_0, the local twistor fibration arising from $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$, $\lambda \neq \lambda'$ breaks down. Note also that Proposition 7.8 also holds for I_4 without any change of the proof, and it implies that if members of $\mathcal{L}_\lambda^+$ (resp. $\mathcal{L}_\lambda^-$) are twistor lines for $\lambda \in I_4^-$, members of $\mathcal{L}_\lambda^-$ (resp. $\mathcal{L}_\lambda^+$) must be twistor lines for $\lambda \in I_4^+$.

7.3. **The case of special type.** In this subsection we calculate the normal bundle of L^+ and L^- in Z, where L^+ and L^- are curves which are mapped biholomorphically onto a real touching conic of special type. Compared to the case of generic type, the problem becomes harder and the result becomes more complicated, since touching conics of special type go through the singular point P_∞ and $\overline{P}_\infty$ of B, so that the situation, and hence the result also, depend on how we resolve the corresponding singularities of Z_0.

First we recall the situation and fix notations. Let $\Phi_0 : Z_0 \to \mathbf{CP}^3$ be the double covering branched along B. Put $p_\infty := \Phi_0^{-1}(P_\infty)$. In a neighborhood of $P_\infty = (0:0:0:1)$, we use (x_0, x_1, x_2) as an affine coordinate by setting $x_i = y_i/y_3$. Then around $P_\infty = (0,0,0)$, B is given by the equation $(x_2 + Q(x_0,x_1))^2 - x_0x_1(x_0+x_1)(x_0 - ax_1) = 0$. Let z be a fiber coordinate of $O(2) \to \mathbf{CP}^3$. Then Z_0 is given by the equation

$$z^2 + (x_2 + Q(x_0,x_1))^2 - x_0x_1(x_0+x_1)(x_0-ax_1) = 0. \tag{41}$$

This can be also written as $\{z + i(x_2 + Q(x_0,x_1))\}\{z - i(x_2 + Q(x_0,x_1))\} = x_0x_1(x_0 + x_1)(x_0 - ax_1)$. Setting $\xi = z + i(x_2 + Q(x_0,x_1))$ and $\eta = z - i(x_2 + Q(x_0,x_1))$, we get

$$Z_0 : \quad \xi\eta = x_0x_1(x_0+x_1)(x_0-ax_1). \tag{42}$$

Thus $p_\infty = \{(x_0,x_1,\xi,\eta) = (0,0,0,0)\}$ is a compound A_3-singularity. Small resolutions of p_∞ are explicitly given by steps as follows: first we choose ordered three linear forms $\{\ell_1,\ell_2,\ell_3\} \subset \{x_0, x_1, x_0+x_1, x_0-ax_1\}$. Next blow-up Z_0 along a surface $\{\xi = \ell_1 = 0\}$. Since this surface is contained in the threefold Z_0, and since it goes through p_∞, the exceptional locus is a smooth rational curve. Concretely, by setting $\xi = u\ell_1$ we get a threefold

$$u\eta = x_0x_1(x_0+x_1)(x_0-ax_1)/\ell_1 \tag{43}$$

which has a compound A_2-singularity at the origin. The exceptional curve is given by $\Gamma_1 := \{(u,\eta,x_0,x_1) \,|\, \eta = x_0 = x_1 = 0\}$. We can use u as an affine coordinate on Γ_1. Next, as the center of the second blow-up, we choose $\{u = \ell_2 = 0\}$ which is also contained in the threefold (43). Setting $u = v\ell_2$, we get

$$v\eta = x_0x_1(x_0+x_1)(x_0-ax_1)/\ell_1\ell_2 \tag{44}$$

which still has an ordinary double point at the origin. The exceptional curve of this second blow-up is given by $\Gamma_2 := \{(v,\eta,x_0,x_1) \,|\, \eta = x_0 = x_1 = 0\}$, on which we can use v as an affine coordinate. Finally by blowing up along $\{v = \ell_3 = 0\}$ and setting $v = w\ell_3$, we get

$$w\eta = x_0x_1(x_0+x_1)(x_0-ax_1)/\ell_1\ell_2\ell_3, \tag{45}$$

which is clearly smooth in a neighborhood of the origin. The exceptional curve of the last small resolution is given by $\Gamma_3 := \{(w,\eta,x_0,x_1) \,|\, \eta = x_0 = x_1 = 0\}$, on which we can use w as an affine coordinate. Thus the composition of these three blow-ups provides a small resolution of p_∞, whose exceptional curve is $\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3$ that is a chain of smooth rational curves. (See Figure 8.)

Note that this small resolution of the singularity p_∞ is easily seen to be $U(1)$-equivariant, since our $U(1)$-action is written as

$$(\xi, \eta, x_0, x_1) \mapsto (e^{2i\theta}\xi, e^{2i\theta}\eta, e^{i\theta}x_0, e^{i\theta}x_1).$$

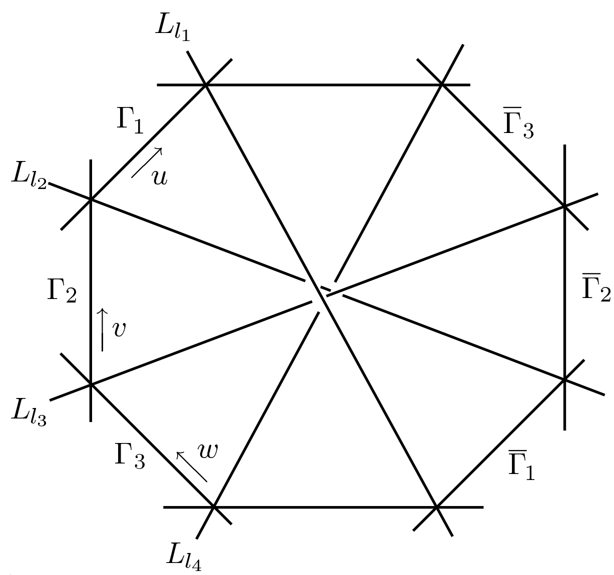

FIGURE 8. the inverse image of l_∞ and four tropes

Once a resolution of p_∞ is given, it naturally determines that of $\overline{p}_\infty$ by reality. Let $\overline{\Gamma} = \overline{\Gamma}_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$ be the exceptional curve over $\overline{p}_\infty$. Let $\mu_\infty : Z_0' \to Z_0$ be the small resolution of p_∞ and $\overline{p}_\infty$ produced in this way (for some choice of ℓ_1, ℓ_2 and ℓ_3). Note that Z_0' yet has a real ordinary double point over the unique real double point of B (which was denoted by P_0). Each choice of ℓ_1, ℓ_2, ℓ_3 (and ℓ_4) determines a small resolution of p_∞ and $\overline{p}_\infty$ and there are $4! = 24$ ways of resolutions in all.

Next we prove the following lemma which is promised in the proof of Proposition 2.7. Let $H_{\ell_i} \subset \mathbf{CP}^3$ $(1 \le i \le 4)$ be a real plane defined by $\ell_i = 0$. Let $T_{\ell_i} \subset H_{\ell_i}$ be the tropes of B. $\Phi^{-1}(T_{\ell_i})$ is defined by $\xi = \eta = \ell_i = 0$ around p_∞. $\Phi_0^{-1}(H_{\ell_i})$ consists of two irreducible components H'_{ℓ_i} and $\overline{H}'_{\ell_i}$, both of which are $\mathbf{C}^*$-equivariantly biholomorphic to H_{ℓ_i} $(\simeq \mathbf{CP}^2)$. (These are defined by $\xi = \ell_i = 0$ and $\eta = \ell_i = 0$ around p_∞. We have $\Phi_0^{-1}(T_{\ell_i}) = H'_{\ell_i} \cap \overline{H}'_{\ell_i}$.) Therefore $\mu_\infty^{-1}(\Phi_0^{-1}(H_{\ell_i}))$ also consists of two irreducible components, which we write D_{ℓ_i} and $\overline{D}_{\ell_i}$.

Lemma 7.9. *Consider a $\mathbf{C}^*$-action on $\mathbf{CP}^2$ defined by $(y_1, y_2, y_3) \mapsto (y_1, ty_2, t^{-1}y_3)$, $t \in \mathbf{C}^*$, which has smooth conics as the closure of general orbits. group of $\mathbf{CP}^2$.) Let C be one of such conics. Choose any one of the two $\mathbf{C}^*$-fixed points on C and let $D \to \mathbf{CP}^2$ be the composition of three blow-ups whose centers are always the $\mathbf{C}^*$-fixed point on (the strict transform of) C just chosen. Then the natural $\mathbf{C}^*$-equivariant morphism $D_{\ell_4} \to H_{\ell_4}$ is equivariantly biholomorphic to the morphism $D \to \mathbf{CP}^2$.*

In particular, Z always contains a $\mathbf{C}^*$-invariant divisor D_{ℓ_4} which is $U(1)$-equivariantly biholomorphic to D, and such that $D_{\ell_4} + \overline{D}_{\ell_4}$ is a fundamental divisor.

Proof of the lemma. First, notice that for any $\mathbf{C}^*$-invariant plane $H \supset l_\infty$, our $\mathbf{C}^*$-action coincides the $\mathbf{C}^*$-action given in the lemma. We prove the claim for D_{ℓ_4} by direct calculation using the local coordinate (ξ, η, x_0, x_1) introduced above. D_{ℓ_4} can be assumed to be the

inverse image (by μ) of the surface $H'_{\ell_4} = \{\eta = \ell_4 = 0\} \subset Z_0$. As is described above, we first blow-up Z_0 along $\{\xi = \ell_1 = 0\}$. Then since we can use (ξ, ℓ_1) as a local coordinate on H'_{ℓ_4} around p_∞(= the origin), H'_{ℓ_4} is actually blown-up at p_∞. Then (the inverse image of) H'_{ℓ_4} is still defined by $\{(u, \eta, x_0, x_1) \,|\, \eta = \ell_4 = 0\}$, where $u = \xi/\ell_1$ is an affine coordinate on the exceptional curve as before. Hence for the second blow-up whose center is $\{u = \ell_2 = 0\}$, the origin of the surface $\{\eta = \ell_4 = 0\}$ is again blown-up. The situation is the same for the third blow-up. Thus the original surface $H'_{\ell_4} \subset Z_0$ is blown-up three times at the $\mathbf{C}^*$-fixed point. Moreover, the blown-up points are always over the trope on H_{ℓ_4}, as is easily verified from the fact that on the surface H'_{ℓ_4}, $\Phi_0^{-1}(T_{\ell_4})$ is given by $\xi = 0$. (Note that our coordinate change is not linear and a conic on H_{ℓ_i} is transformed into a 'line' on H'_{ℓ_4} with respect to the coordinate (ξ, ℓ_1).)

So far we have only seen what happens around p_∞. Around $\overline{p}_\infty$, we claim that on the surface H'_{ℓ_4} nothing change through our blow-ups. To see this, set $x'_i = y_i/y_2$ for $i = 0, 1, 3$. Then in a neighborhood of $\overline{p}_\infty$, Z_0 is defined by

$$z^2 + (x'_3 + Q(x'_0, x'_1))^2 - x'_0 x'_1 (x'_0 + x'_1)(x'_0 - a x'_1) = 0.$$

Thus setting $\xi' = z + i(x'_3 + Q(x'_0, x'_1))$ and $\eta' = z - i(x'_3 + Q(x'_0, x'_1))$, we can write

$$Z_0 : \quad \xi' \eta' = x'_0 x'_1 (x'_0 + x'_1)(x'_0 - a x'_1).$$

Then since we have $\sigma^* z = \overline{z}$, $\sigma^* x_0 = x'_0$, $\sigma x_1 = \overline{x}'_1$ and $\sigma^* x_2 = \overline{x}'_3$, we have

$$\begin{aligned}\sigma^* \xi &= \sigma^* \left(z + i(x_2 + Q(x_0, x_1))\right) = \overline{z} + i(\overline{x}_3 + Q(\overline{x}_0, \overline{x}_1)) \\ &= \overline{z} + i(\overline{x}_3 + Q(\overline{x}_0, \overline{x}_1)) = \overline{z - i(x_3 + Q(x_0, x_1))} = \overline{\eta}'.\end{aligned}$$

Similarly we have $\sigma^* \eta = \overline{\xi}'$. On the other hand, we easily get $\sigma^* \ell_i = \overline{\ell}_i$ for $1 \le i \le 4$. Hence $\overline{H}'_{\ell_4} = \sigma(H'_{\ell_4})$ is defined by $\xi' = \ell_4 = 0$ in a neighborhood of $\overline{p}_\infty$. This implies that H'_{ℓ_4} is defined by $\eta' = \ell_4 = 0$ around $\overline{p}_\infty$. On the other hand, by reality, the center of the first blow-up must be $\{\eta' = \ell_1 = 0\}$, which intersect H'_4 transversally along a smooth curve. Therefore, H'_{ℓ_4} has no effect under the first blow-up of Z_0. Similarly, nothing happens for the remaining two blow-ups. Combined with what we have seen around p_∞, we have shown that $D_{\ell_4} \to H_{\ell_4}$ is $\mathbf{C}^*$-equivariantly biholomorphic to D in the lemma. $\square$

By similar direct calculations, we can show the following

Lemma 7.10. *Let $\mu_\infty^{-1}(\Phi_0^{-1}(H_{\ell_i})) = D_{\ell_i} + \overline{D}_{\ell_i}$ and $T_{\ell_i} \subset H_{\ell_i}$ $(1 \le i \le 4)$ be as above. Then $L_{\ell_i} = D_{\ell_i} \cap \overline{D}_{\ell_i}$ is a real smooth rational curve in Z'_0 satisfying the following properties: (i) L_{ℓ_i} are mapped biholomorphically onto T_{ℓ_i}, (ii) the normal bundle of L_{ℓ_i} in Z'_0 is isomorphic to $O(1)^{\oplus 2}$, (iii) the intersection of L_{ℓ_i} with $\Phi^{-1}(l_\infty)$ is as in Figure 8.*

Sketch of the proof. We use the local coordinates in the previous proof. $\Phi_0^{-1}(H_{\ell_i})$ consists of two irreducible components H'_{ℓ_i} and $\overline{H}'_{\ell_i}$. These are $\mathbf{C}^*$-invariant surfaces. Since the trope T_{ℓ_i} is a conic on H_{ℓ_i}, the normal bundle of the intersection $H'_{\ell_i} \cap \overline{H}'_{\ell_i}$ in H'_{ℓ_i} and $\overline{H}'_{\ell_i}$ are isomorphic to $O(4)$. We may suppose that H'_{ℓ_i} is defined by $\eta = \ell_i = 0$ as in the previous proof. Then if we blow-up along $\xi = \ell_1 = 0$, H'_{ℓ_i} is actually blown-up at p_∞ for $2 \le i \le 4$, whereas nothing happens for H'_{ℓ_1}. On the other hand, around $\overline{p}_\infty$, nothing happens on H'_{ℓ_i} for $2 \le i \le 4$, whereas H'_{ℓ_1} is blown-up at $\overline{p}_\infty$. (Here note that p_∞ and $\overline{p}_\infty$ are just the fixed locus of $\mathbf{C}^*$-action on $H'_{\ell_i} \cap \overline{H}'_{\ell_i}$.) Consequently, the resulting space has two divisors H''_{ℓ_i} and $\overline{H}''_{\ell_i}$ that are the inverse images of H'_{ℓ_i} and $\overline{H}'_{\ell_1}$ respectively. The intersection $H''_{\ell_i} \cap \overline{H}''_{\ell_i}$ is a smooth $\mathbf{C}^*$-invariant rational curves whose normal bundles in H''_{ℓ_i} and $\overline{H}''_{\ell_i}$ are $O(3)$. For the second blow-ups, H''_{ℓ_i}, $i = 1, 3, 4$ (resp. $\overline{H}''_{\ell_i}$, $i = 1, 3, 4$), are blown-up at the $\mathbf{C}^*$-fixed point on $H''_{\ell_i} \cap \overline{H}''_{\ell_i}$ lying over p_∞ (resp. $\overline{p}_\infty$), and H''_{ℓ_2} (resp. $\overline{H}''_{\ell_2}$) is blown-up at the $\mathbf{C}^*$-fixed

point on $H''_{\ell_i} \cap \overline{H}''_{\ell_i}$ lying over $\overline{p}_\infty$ (resp. p_∞). Denoting $H^{(3)}_{\ell_i}$ and $\overline{H}^{(3)}_{\ell_i}$ the inverse image of H''_{ℓ_i} and $\overline{H}''_{\ell_i}$ respectively, the normal bundles of $H^{(3)}_{\ell_i} \cap \overline{H}^{(3)}_{\ell_i}$ in $H^{(3)}_{\ell_i}$ and $\overline{H}^{(3)}_{\ell_i}$ become $O(2)$. The third (=the last) blow-up yields D_{ℓ_i} and $\overline{D}_{\ell_i}$ as the inverse images of $H^{(3)}_{\ell_i}$ and $\overline{H}^{(3)}_{\ell_i}$ respectively. $D_{\ell_i} \to H^{(3)}_{\ell_i}$, $i = 1, 2, 4$ (resp. $\overline{D}_{\ell_i} \to \overline{H}^{(3)}_{\ell_i}$, $i = 1, 2, 4$) is the blow-up at the $\mathbf{C}^*$-fixed point of $H^{(3)}_{\ell_i} \cap \overline{H}^{(3)}_{\ell_i}$ lying over p_∞ (resp. $\overline{p}_\infty$), whereas $D_{\ell_3} \to H^{(3)}_{\ell_3}$(resp. $\overline{D}_{\ell_3} \to \overline{H}^{(3)}_{\ell_i}$) is the blow-up at the $U(1)$-fixed point lying over $\overline{p}_\infty$ (resp. p_∞). Hence the normal bundles of $L_{\ell_i} = D_{\ell_i} \cap \overline{D}_{\ell_i}$ in D_{ℓ_i} and $\overline{D}_{\ell_i}$ become isomorphic to $O(1)$. Moreover, the intersection of D_{ℓ_i} and $\overline{D}_{\ell_i}$ in Z'_0 is transversal. Therefore we obtain $N_{L_{\ell_i}/Z'_0} \simeq O(1)^{\oplus 2}$. Thus we get (i) and (ii) of the lemma. (iii) is rather easily verified by direct calculations using the local coordinates above, and we omit the detail. (The case L_{ℓ_4} is actually proved in the last lemma.) □

Next we prove that, as promised in the proof of Proposition 3.5, our small resolution μ_∞ gives a resolution of the surface $\Phi_0^{-1}(H_\lambda)$ which has p_∞ and $\overline{p}_\infty$ as its A_3-singularities:

Lemma 7.11. *If $\lambda \neq -1, 0, a, \infty$, $S_\lambda = \Phi^{-1}(H_\lambda)$ is non-singular.*

Proof. This can be also seen by direct calculation using coordinate (ξ, η, x_0, x_1) around p_∞ above. As H_λ is defined by $x_0 = \lambda x_1$, $\Phi_0^{-1}(H_\lambda)$ is locally defined by $\xi\eta = f(\lambda)x_1^4$ around p_∞. Substituting $\xi = u\ell_1$ for the first blow-up, the inverse image becomes $u\eta = (f(\lambda)/\ell_1(\lambda, 1))x_1^3$. Next substituting $u = v\ell_2$ for the second blow-up, we get $v\eta = (f(\lambda)/\ell_1(\lambda, 1)\ell_2(\lambda, 1))x_1^2$. Finally substituting $v = w\ell_3$ for the third blow-up, we get $w\eta = (f(\lambda)/\ell_1(\lambda, 1)\ell_2(\lambda, 1)\ell_3(\lambda, 1))x_1$. This is smooth. By reality, the conjugate singular point $\overline{p}_\infty$ of S_λ is also resolved by our blow-ups. Thus both p_∞ and $\overline{p}_\infty$ are resolved. These are of course minimal resolution of the A_3-singularity and the self-intersection number of each irreducible component in S_λ is -2.

If $\lambda \neq \lambda_0$ (and $\lambda \neq -1, 0, a, \infty$), $B \cap H_\lambda$ has no singular point other than p_∞ and $\overline{p}_\infty$, and it follows that S_λ is smooth provided $\lambda \neq \lambda_0$. Because B has a real ordinary double point at $P_0 = (\lambda_0 : 1 : 0 : 0)$, $\Phi_0^{-1}(H_{\lambda_0})$ has a real ordinary double point over P_0. But this singular point is also resolved through small resolution of Z_0. This is of course rather simpler than the above case and we omit the calculation. □

Next in order to calculate the intersection L^+ and L^- (cf. the beginning of this subsection) with Γ, we need a one-parameter presentation of C_θ, in a neighborhood of p_∞:

Lemma 7.12. *Let $C_\theta \subset H_\lambda$ be a real touching conic of special type whose equation is given by (18), and (x_0, x_1, x_2) the affine coordinate around P_∞ as above. Then in a neighborhood of P_∞, C_θ has a one-parameter presentation of the following form:*

$$\tag{46} \begin{cases} x_0 = \lambda x_1 \\ x_2 = -Be^{-i\theta}x_1 - \dfrac{\sqrt{Q^2 - f} + Q}{2}x_1^2 + \dfrac{\sqrt{Q^2 - f} + Q}{2}Be^{i\theta}x_1^3 + O(x_1^4), \end{cases}$$

where we put

$$B := B(\lambda) = \left(\frac{\sqrt{Q^2 - f} - Q}{2}\right)^{\frac{1}{2}}.$$

Note again that $f < 0$ guarantees $\sqrt{Q^2 - f} - Q > 0$.

Proof. By solving (18) with respect to x_2, we get

$$\tag{47} x_2 = -g(x_1) \cdot x_1, \quad g(x_1) := \frac{Be^{-i\theta} + \sqrt{Q^2 - f}\ x_1}{1 + Be^{i\theta}x_1}.$$

Calculating the Maclaurin expansion of $g(x_1)$, we get (46). This is a routine work and we omit the detail. $\square$

Lemma 7.13. *In a neighborhood of p_∞, each of the two irreducible components of $\Phi_0^{-1}(C_\theta)$ has a one-parameter presentation with respect to x_1 in the following forms respectively:*

$$\xi = -2iBe^{-i\theta}x_1 + O(x_1^2), \quad \eta = \frac{ie^{i\theta}f}{2B}x_1^3 + O(x_1^4), \quad x_0 = \lambda x_1, \tag{48}$$

and

$$\xi = -\frac{ie^{i\theta}f}{2B}x_1^3 + O(x_1^4), \quad \eta = 2iBx_1e^{-i\theta} + O(x_1^2), \quad x_0 = \lambda x_1. \tag{49}$$

Proof. First by substituting $x_0 = \lambda x_1$ into (41), we get

$$z^2 = (f - Q^2)x_1^4 - 2Qx_1^2x_2 - x_2^2.$$

Substituting (47) into this, we get

$$z^2 = \left\{(f - Q^2)x_1^2 + 2Qg(x_1)x_1 - g(x_1)^2\right\} x_1^2.$$

Hence we have

$$z = \pm k(x_1)\, x_1, \quad k(x_1) = \left\{(f - Q^2)x_1^2 + 2Qg(x_1)x_1 - g(x_1)^2\right\}^{\frac{1}{2}}.$$

From this we deduce

$$\xi = z + i(x_2 + Qx_1^2) = (\pm k(x_1) - ig(x_1))\; x_1 + iQx_1^2$$

and

$$\eta = z - i(x_2 + Qx_1^2) = (\pm k(x_1) + ig(x_1))\; x_1 - iQx_1^2.$$

Then we get the desired equations by calculating the Maclaurin expansions of $\pm k(x_1) - ig(x_1)$ and $\pm k(x_1) + ig(x_1)$. These are also routine works and we omit the detail. $\square$

Lemma 7.14. *Let L_θ^+ and L_θ^- be the curves in Z_0' which are the proper transforms of the curves (48) and (49) respectively. Then we have: (i) L_θ^+ and Γ_1 intersect transversally at a unique point satisfying*

$$u = -2iBe^{-i\theta} \cdot \frac{x_1}{\ell_1} \tag{50}$$

and $L_\theta^+ \cap \Gamma_2$ and $L_\theta^+ \cap \Gamma_3$ are empty, (ii) $L_\theta^- \cap \Gamma_1$ and $L_\theta^- \cap \Gamma_2$ are empty and L_θ^- and Γ_3 intersect transversally at a unique point satisfying

$$w = -\frac{ie^{i\theta}f}{2B} \cdot \frac{x_1^3}{\ell_1\ell_2\ell_3} \tag{51}$$

where we use u and w as local coordinates on Γ_1 and Γ_3 respectively as explained before, and $B = B(\lambda)$ is as in Lemma 7.12.

Here note that x_1/ℓ_1 and $x_1^3/\ell_1\ell_2\ell_3$ do not depend on x_1, and depend on λ only. Further, we have $B > 0$ since $f < 0$.

Proof. By substituting $\xi = u\ell_1$ into (48), we get the inverse image of (48) to be

$$u\ell_1 = -2iBe^{-i\theta}x_1 + O(x_1^2), \quad \eta = \frac{ie^{i\theta}f}{2B}x_1^3 + O(x_1^4), \quad x_0 = \lambda x_1.$$

Excluding the equation of $\Gamma_1 = \{\eta = x_0 = x_1 = 0\}$, we get the equation of the proper transform in the threefold (43) to be

$$u = -2iBe^{-i\theta}\frac{x_1}{\ell_1} + O(x_1), \quad \eta = \frac{ie^{i\theta}f}{2B}x_1^3 + O(x_1^4), \quad x_0 = \lambda x_1.$$

By setting $x_1 = 0$, we get $u = -2iBe^{-i\theta} \cdot x_1/\ell_1$, and the intersection (of Γ_1 and L^+_θ) is transversal. Finally as remarked above, B is non-zero. Therefore the remaining two blow-ups do not have effect on the intersection. Hence we get (50). Similar calculations show (51). (Note that we have $\xi = w\,\ell_1\ell_2\ell_3$.) $\square$

As in the previous subsection we put $\mathcal{L}^+_\lambda = \{L^+_\theta \,|\, \theta \in \mathbf{R}\}$ and $\mathcal{L}^-_\lambda = \{L^-_\theta \,|\, \theta \in \mathbf{R}\}$. (Note that this time we explicitly specified L^+_θ and L^-_θ respectively in Lemma 7.13.) $U(1)$ again acts transitively on the parameter spaces of these families. By Lemma 7.14, $L^+_\theta \cap \Gamma_1$ is a point, and $\{L^+_\theta \cap \Gamma_1 \,|\, \theta \in \mathbf{R}\}$ is a circle in Γ_1 whose radius is

$$h_1(\lambda) := 2B \cdot |x_1/\ell_1|.$$

Similarly, $\{L^-_\theta \cap \Gamma_3 \,|\, \theta \in \mathbf{R}\}$ is a circle in Γ_3 whose radius is

$$h_3(\lambda) := (-f/2B) \cdot |x_1^3/\ell_1\ell_2\ell_3|.$$

(h_2 will appear in the next subsection.) Then we have the following proposition, which implies that the normal bundles of L^+_θ and L^-_θ in Z'_0 are determined by the behavior of h_1 and h_3 respectively.

Proposition 7.15. *Assume $\lambda \in I_1 \cup I_3$ and take any $L = L^+_\theta \in \mathcal{L}^+_\lambda$. Then either $N_{L/Z'_0} \simeq O(1)^{\oplus 2}$ or $N_{L/Z'_0} \simeq O \oplus O(2)$ holds. Further, the latter holds iff λ is a critical point of h_1 above. The same claims also hold for $\mathcal{L}^-_\lambda$ if we replace h_1 by h_3.*

Proof. The first claim can be proved in the same way as in Proposition 7.3. The other claims can also be proved in the same manner as in Proposition 7.3: take any $L \in \mathcal{L}^+_\lambda$. Then the two one-parameter families of L in Z'_0 in the previous proof make senses also in this case, so that we again have two linearly independent sections s and t of N_λ, $N_\lambda = N_{L/Z'_0}$. Then the previous proof works if we replace h_0 by h_1, $\Phi^{-1}(l_\infty)$ by $\Gamma_1 \cup \overline{\Gamma}_1$, and (35) by (50). For $L \in \mathcal{L}^-_\lambda$, replace h_0 by h_3, $\Phi^{-1}(l_\infty)$ by $\Gamma_3 \cup \overline{\Gamma}_3$, and (35) by (51). $\square$

By definition, h_1 and h_3 depend on the choice of ℓ_1, ℓ_2 and ℓ_3. Therefore, Proposition 7.15 implies that the normal bundles of L^+_θ and L^-_θ in Z'_0 depend on how we resolve p_∞. More precisely, the normal bundle of L^+_θ depends on the choice of ℓ_1 only, whereas the normal bundle of L^-_θ depends on that of $\{\ell_1, \ell_2, \ell_3\}$.

Thus we need to know the critical points of h_1 and h_3 for every choices of ℓ_1, ℓ_2 and ℓ_3. At first glance there may seem to be too many functions to be investigated, but it is easily seen that h_3 is a reciprocal of h_1 (up to a constant) for some other choice of ℓ_1, ℓ_2 and ℓ_3. Consequently, what we need to know is the behavior of h_1 for the four choices of ℓ_1. (Behavior of these functions near the endpoints of I_1 and I_3 below will be needed in §7.5.)

Lemma 7.16. *(i) If $\ell_1 = x_1$, h_1 has no critical point on I_1, and has a unique critical point on I_3. Further, we have* $\lim_{\lambda\downarrow-\infty} h_1(\lambda) = +\infty$ *and* $h_1(-1) = 0$. *(ii) If $\ell_1 = x_0$, h_1 has a unique critical point on I_1, and no critical point on I_3. Further, we have* $\lim_{\lambda\downarrow 0} h_1(\lambda) = +\infty$ *and* $h_1(a) = 0$. *(iii) If $\ell_1 = x_0 + x_1$, h_1 has no critical point on I_1, and has a unique critical point on I_3. Further, we have* $\lim_{\lambda\downarrow-\infty} h_1(\lambda) = 0$ *and* $\lim_{\lambda\uparrow-1} h_1(\lambda) = +\infty$. *(iv) If $\ell_1 = x_0 - ax_1$, h_1 has a unique critical point on I_1, and no critical point on I_3. Further, we have* $h_1(0) = 0$ *and* $\lim_{\lambda\uparrow a} h_1(\lambda) = +\infty$. *(See Figure 9.)*

Proof. (i) If $\ell_1 = x_1$, we have $h_1^2 = 2(\sqrt{Q^2 - f} - Q)$. Since $h_1 > 0$ on $I_1 \cup I_3$, the critical points of h_1^2 and h_1 coincide on $I_1 \cup I_3$. We think of h_1^2 as a real valued function defined on the whole of $\mathbf{R}$, but which is not differentiable at $\lambda = \lambda_0$ in general. It is immediate to see that $h_1^2(-1) = h_1^2(0) = h_1^2(a) = 0$, $\lim_{\lambda\downarrow-\infty} h_1^2(\lambda) = +\infty$ and $\lim_{\lambda\uparrow\infty} h_1^2(\lambda) = -\infty$. Then because h_1^2 is differentiable on $\lambda \neq \lambda_0$, h_1^2 has a critical point on I_2 and I_3 respectively. On

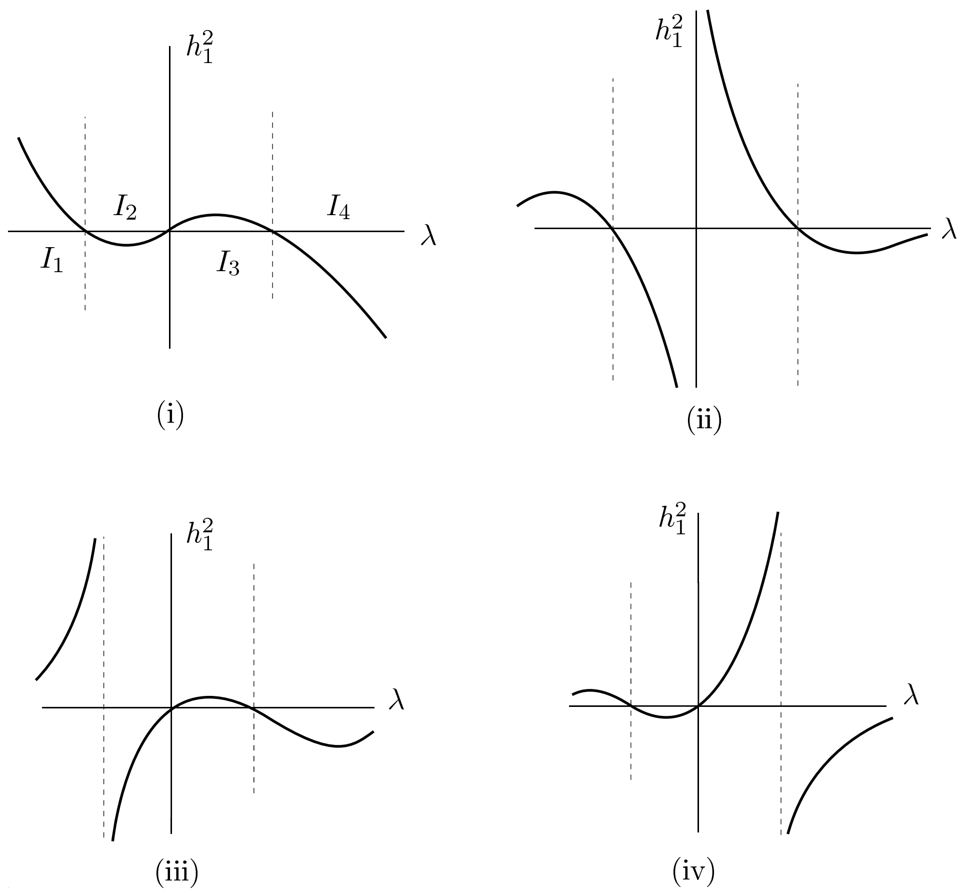

FIGURE 9. behaviors of h_1^2

the other hand, we have

$$\left(\sqrt{Q^2-f}-Q\right)' = \frac{2QQ'-f'-2Q'\sqrt{Q^2-f}}{2\sqrt{Q^2-f}},$$

and it follows that $(\sqrt{Q^2-f}-Q)'=0$ implies

$$(2QQ'-f')^2 = 4Q'^2(Q^2-f). \tag{52}$$

It is readily seen that the degree of both hand sides of (52) are six, and that both have $(\lambda-\lambda_0)^2$ as a factor. Since we have already got two critical points of h_1^2 other than $\lambda=\lambda_0$, there are at most two solutions of (52) remaining.

We set $g := 2(-\sqrt{Q^2-f}-Q)$ which is also defined on $\mathbf{R}$ and possibly not differentiable at $\lambda=\lambda_0$. Note that if we replace h_1^2 by g on $\lambda\geq\lambda_0$, then the resulting function is differentiable at $\lambda=\lambda_0$. It is easily verified that $g'=0$ also implies (52) and it gives a solution not coming from $(h_1^2)'=0$. Further, we readily have $\lim_{\lambda\downarrow-\infty} g(\lambda) = \lim_{\lambda\uparrow\infty} g(\lambda) = -\infty$.

Suppose that g has a critical point. Then together with the above two critical points of h_1^2 on $I_2\cup I_3$, we have three solutions of (52) other than $\lambda=\lambda_0$. If h_1^2 has critical points on I_1, its number is at least two. This implies that (52) has five solutions other than $\lambda=\lambda_0$ and this is a contradiction. Therefore h_1^2 has no critical points on I_1, if g has a critical point. Similarly, if the number of the critical points on I_3 is not one, then it must be at least three. This is again a contradiction. Thus if g has a critical point, h_1^2, and hence h_1 has no critical point on I_1 and a unique critical point on I_3.

So suppose that g has no critical point. This happens exactly when g attains the maximal value at $\lambda = \lambda_0$. Then we have $\lim_{\lambda \uparrow \lambda_0} g'(\lambda) > 0$, since otherwise g has a critical point on $\lambda < \lambda_0$. Because we have $\lim_{\lambda \uparrow \lambda_0} g'(\lambda) = \lim_{\lambda \downarrow \lambda_0} (h_1^2)'(\lambda)$, we get $\lim_{\lambda \downarrow \lambda_0} (h_1^2)'(\lambda) > 0$. Since $\lim_{\lambda \uparrow \infty} h_1^2(\lambda) = -\infty$, it follows that h_1^2 has a critical point on I_4. Thus we get three solutions of (52) other than λ_0. Then the same argument in the case that g has a critical point as above, we can deduce that h_1 has no critical point on I_1 and a unique critical point on I_3. Thus we get the claim of (i) concerning critical points of h_1. The remaining claims of (i) immediately follows from the definition of h_1.

Claims of (ii), (iii) and (iv) about critical points can be obtained by applying a projective transformation $\lambda \mapsto 1/\lambda$ for the case (ii), $\lambda \mapsto 1/(\lambda+1)$ for the case (iii), and $\lambda \mapsto 1/(\lambda - a)$ for the case (iv) respectively. The other claims are immediate to see. $\square$

As is already mentioned, the behavior of h_3 can be easily seen from that of h_1 for some other choice of ℓ_1, ℓ_2 and ℓ_3. The result is the following:

Lemma 7.17. *(i) If $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_0 + x_1, x_0 - ax_1\}$, h_3 has no critical point on I_1, and has a unique critical point on I_3. Further, we have $\lim_{\lambda \downarrow -\infty} h_3(\lambda) = 0$ and $\lim_{\lambda \uparrow -1} h_3(\lambda) = +\infty$. (ii) If $\{\ell_1, \ell_2, \ell_3\} = \{x_1, x_0 + x_1, x_0 - ax_1\}$, h_3 has a unique critical point on I_1, and no critical point on I_3. Further, we have $h_3(0) = 0$ and $\lim_{\lambda \uparrow a} h_3(\lambda) = +\infty$. (iii) If $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, x_0 - ax_1\}$, h_3 has no critical point on I_1, and has a unique critical point on I_3. Further, we have $\lim_{\lambda \downarrow -\infty} h_3(\lambda) = +\infty$ and $h_3(-1) = 0$. (iv) If $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, x_0 + x_1\}$, h_3 has a unique critical point on I_1, and no critical point on I_3. Further, we have $\lim_{\lambda \downarrow 0} h_3(\lambda) = +\infty$ and $h_3(a) = 0$.*

Note that in the lemma we do not specify the ordering of ℓ_1, ℓ_2 and ℓ_3; for instance $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_0 + x_1, x_0 - ax_1\}$ in (i) does not imply $\ell_1 = x_0, \ell_2 = x_0 + x_1$ and $\ell_3 = x_0 - ax_1$.

Corollary 7.18. *For any choice of ℓ_1, ℓ_2 and ℓ_3, the following (i) and (ii) hold: (i) members of $\mathcal{L}_\lambda^+$ ($\lambda \in I_1$) and $\mathcal{L}_\lambda^+$ ($\lambda \in I_3$) cannot be twistor lines at the same time, (ii) the same claim holds also for $\mathcal{L}_\lambda^-$.*

Proof. Suppose that $\lambda \in I_1 \cup I_3$ is a critical point of h_1. Then by Proposition 7.15, any member of $\mathcal{L}_\lambda^+$ is not a twistor line because its normal bundle in Z_0' is $O \oplus O(2)$. Then just as in the proof of Proposition 7.5, any member of $\mathcal{L}_\mu^+$ cannot be a twistor line provided that $\mu \in I_1 \cup I_3$ and λ belong to the same interval (I_1 or I_3). By Lemma 7.16, h_1 necessarily has a critical point on just one of I_1 and I_3. Hence (i) holds. The proof is the same for $\mathcal{L}_\lambda^-$ if we use Lemma 7.17 instead. $\square$

Thus together with Proposition 7.15, we have obtained new families of real smooth rational curves which have $O(1)^{\oplus 2}$ as their normal bundles, but which are not twistor lines.

Proposition 7.15 and Lemmas 7.16 and 7.17 enable us to determine the normal bundles of L_θ^+ and L_θ^- in Z_0' for every choices of small resolutions of p_∞. In particular, the normal bundles of L_θ^+ and L_θ^- in Z_0', and also which component has to be chosen as candidates of twistor lines, depend on the choice made.

7.4. **The case of orbit type.** Suppose $\lambda \in I_2 \cup I_4$. In this subsection we calculate the normal bundles of L_α^+ and L_α^- in Z_0', where L_α^+ and L_α^- are curves which are mapped biholomorphically onto a real touching conic $C_\alpha \in \mathcal{C}_\lambda^{\rm orb}$ defined by (25). Note again that C_α and $L_\alpha^\pm$ depend not only on α, but also on $\lambda \in I_2 \cup I_4$. Compared to generic type and special type, calculations are much easier since the equations of touching conics of orbit type are much simpler.

First we make a distinction of L_α^+ and L_α^-. We use local coordinates (x_0, x_1, x_2, z) and (x_0, x_1, ξ, η) as in the previous subsection. Recall that $\Phi_0^{-1}(H_\lambda)$ is defined by $\xi\eta = f x_1^4$, and

that the equation of irreducible components of $\Phi_0^{-1}(C_\alpha)$ is given by $z = \pm(f-(\alpha+Q)^2)^{\frac{1}{2}}\,x_1^2$ ((34)). Then we denote by L_α^+ (resp. L_α^-) the components corresponding to $z = (f-(\alpha+Q)^2)^{\frac{1}{2}}\,x_1^2$ (resp. $z = -(f-(\alpha+Q)^2)^{\frac{1}{2}}\,x_1^2$). L_α^+ and L_α^- are curves in Z_0'. Of course we have L_α^+ and L_α^- coincide if $\alpha = -Q \pm \sqrt{f}$.

Recall that in the previous subsection we have introduced an affine coordinate $v = \xi/\ell_1\ell_2$ on the exceptional curve Γ_2. Points on Γ_2 are indicated by using this v.

Lemma 7.19. *Let L_α^+ and L_α^- be as above. Then $L_\alpha^+\cap\Gamma_1, L_\alpha^+\cap\Gamma_3, L_\alpha^-\cap\Gamma_1$ and $L_\alpha^-\cap\Gamma_3$ are empty, and $L_\alpha^+\cap\Gamma_2$ and $L_\alpha^-\cap\Gamma_2$ are points satisfying respectively*

$$L_\alpha^+\cap\Gamma_2 = \left\{v = \left(\sqrt{f-(\alpha+Q)^2} + i(\alpha+Q)\right)\frac{x_1^2}{\ell_1\ell_2}\right\}$$

and

$$L_\alpha^-\cap\Gamma_2 = \left\{v = \left(-\sqrt{f-(\alpha+Q)^2} + i(\alpha+Q)\right)\frac{x_1^2}{\ell_1\ell_2}\right\},$$

where each intersection is transversal. Moreover, L_α^+ and L_α^- do not intersect provided $\alpha \neq -Q \pm \sqrt{f}$ (namely $L_\alpha^+ \neq L_\alpha^-$).

Here, note again that $x_1^2/\ell_1\ell_2$ does not depend on x_1.

Proof. Substituting $x_2 = \alpha x_1^2$, we have

$$\xi = z + i(x_2 + Qx_1^2) = \left\{\pm\sqrt{f-(\alpha+Q)^2} + i(\alpha+Q)\right\}x_1^2$$

and

$$\eta = z - i(x_2 + Qx_1^2) = \left\{\pm\sqrt{f-(\alpha+Q)^2} - i(\alpha+Q)\right\}x_1^2$$

over C_α. ($\pm$ corresponds to $L_\alpha^\pm$.) From these and from the explicit resolutions of the previous subsection, we can easily see that for any choice of ℓ_1, ℓ_2 and ℓ_3, $L_\alpha^\pm\cap\Gamma_1$ and $L_\alpha^\pm\cap\Gamma_3$ are empty and that $L_\alpha^+\cap\Gamma_2$ and $L_\alpha^-\cap\Gamma_2$ are points satisfying

$$v = \xi/\ell_1\ell_2 = \left\{\pm\sqrt{f-(\alpha+Q)^2} + i(\alpha+Q)\right\}\frac{x_1^2}{\ell_1\ell_2},$$

where $\pm$ corresponds to L_α^+ and L_α^- respectively. The transversality is evident from this representation. Finally, suppose that $\alpha \neq -Q \pm \sqrt{f}$. Then since C_α intersect B only at P_∞ and $\overline{P}_\infty$, L_α^+ and L_α^- intersect at most on $\Gamma\cup\overline{\Gamma}$. But this does not happen because $f-(\alpha+Q)^2$ is non-zero and hence the values of v we have already got are different. Thus we have obtained all of the claims of the lemma. □

Since $L_\alpha^\pm$ and Γ_2 are $U(1)$-invariant, $L_\alpha\cap\Gamma_2$ must be $U(1)$-fixed point. In particular, any points on Γ_2 is $U(1)$-fixed. From these lemmas, we immediately get the following

Lemma 7.20. *Fix $\lambda \in I_2\cup I_4$. Then the set $\{(L_\alpha^+\cup L_\alpha^-)\cap\Gamma_2 \mid -Q-\sqrt{f} \le \alpha \le -Q+\sqrt{f}\}$ is a circle in Γ_2 whose center is $\Gamma_2\cap\Gamma_3$ $(=\{v=0\})$ and whose radius is $\sqrt{f}\,|x_1^2/\ell_1\ell_2|$.*

The following proposition, which corresponds to Propositions 7.5 (generic type) and 7.15 (special type), can be proved by using the same idea as in Proposition 7.15. So we omit the proof.

Proposition 7.21. *Set $h_2(\lambda) = \sqrt{f}\,(x_1^2/\ell_1\ell_2)$, which is clearly differentiable on $I_2\cup I_4$. Let N denote the normal bundle of L_α^+ in Z_0'. Then we have either $N \simeq O(1)^{\oplus 2}$ or $N \simeq O\oplus O(2)$, and the latter holds iff λ is a critical point of h_2. The same claim holds also for L_α^-.*

Needless to say, h_2 depends on the choice of ℓ_1 and ℓ_2. Thus as in the case of special type, the normal bundles of L_α^+ and L_α^- depend on the choice of small resolution of p_∞. In view of Proposition 7.21, we need to know the critical point of h_2 for each choice of $\{\ell_1, \ell_2\}$.

There are $4!/(2!2!) = 6$ choices of $\{\ell_1, \ell_2\}$. If we take $\{\ell_1, \ell_2\} = \{x_0, x_1\}$ for instance, we have $h_2(\lambda)^2 = (\lambda+1)(\lambda-a)/\lambda$, and it is elementary to determine the critical points of this function. For any other choices, we always get h_2 in explicit form and it is easy to determine their critical points. So here we only present the result:

Lemma 7.22. *(i) If $\{\ell_1, \ell_2\} = \{x_0, x_1\}$, h_2 has no critical point on $I_2 \cup I_4$. Further, $h_2(-1) = 0, \lim_{\lambda\uparrow 0} h_2(\lambda) = +\infty, h_2(a) = 0$ and $\lim_{\lambda\uparrow\infty} h_2(\lambda) = +\infty$. (ii) If $\{\ell_1, \ell_2\} = \{x_0+x_1, x_0-ax_1\}$, h_2 has no critical point on $I_2 \cup I_4$. Further, $\lim_{\lambda\downarrow -1} h_2(\lambda) = +\infty, h_2(0) = 0, \lim_{\lambda\downarrow a} h_2(\lambda) = +\infty$ and $\lim_{\lambda\uparrow\infty} h_2(\lambda) = 0$. (iii) If $\{\ell_1, \ell_2\} = \{x_1, x_0+x_1\}$, h_2 has no critical point on $I_2 \cup I_4$. Further, $\lim_{\lambda\downarrow -1} h_2(\lambda) = +\infty, h_2(0) = 0, h_2(a) = 0$ and $\lim_{\lambda\uparrow\infty} h_2(\lambda) = \infty$. (iv) If $\{\ell_1, \ell_2\} = \{x_0, x_0 - ax_1\}$, h_2 has no critical point on $I_2 \cup I_4$. Further, $h_2(-1) = 0, \lim_{\lambda\uparrow 0} h_2(\lambda) = +\infty, \lim_{\lambda\downarrow a} h_2(\lambda) = +\infty$. (v) If $\{\ell_1, \ell_2\} = \{x_0, x_0+x_1\}$, or if $\{\ell_1, \ell_2\} = \{x_1, x_0 - ax_1\}$, h_2 has a unique critical point on I_2 and I_4 respectively. (See Figure 10.)*

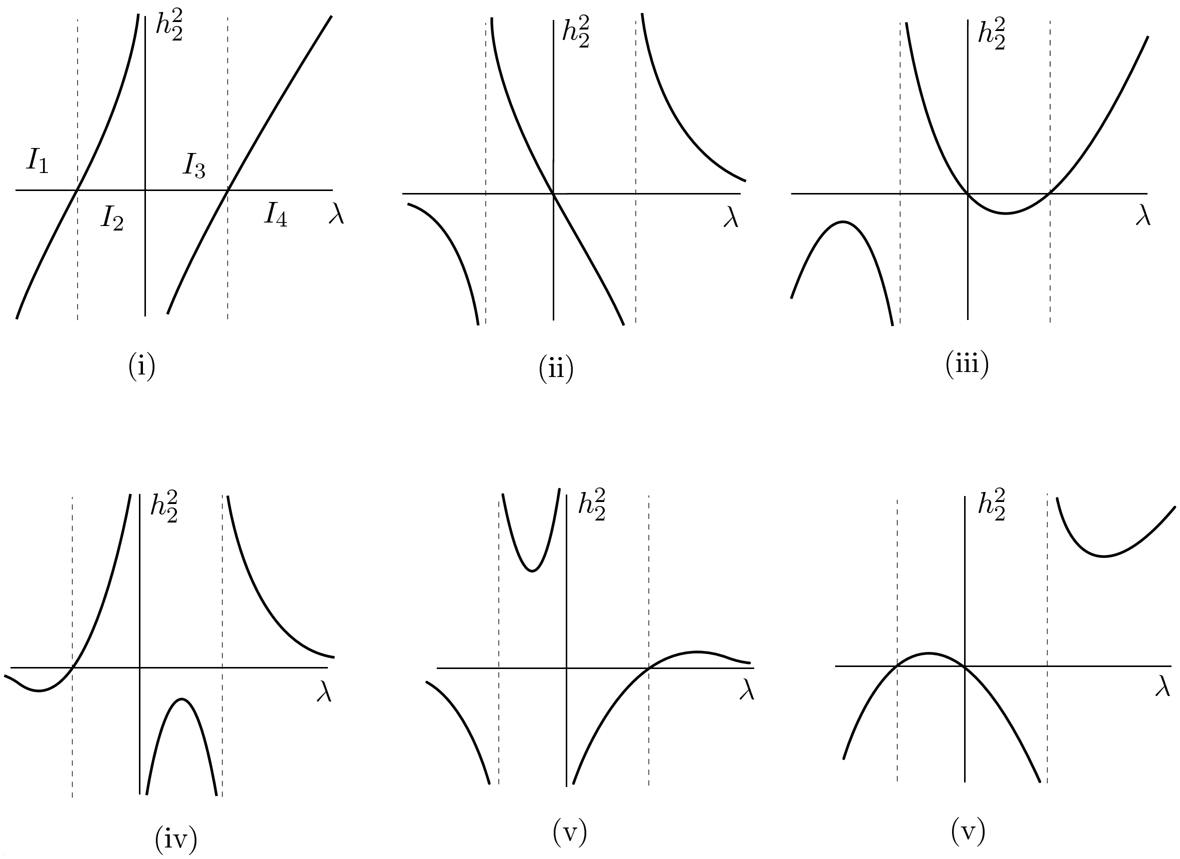

FIGURE 10. behaviors of h_2^2

By Corollary 7.6, if $\lambda \in I_2$, images of twistor lines in H_λ must be of orbit type. Therefore by Proposition 7.21, if a small resolution of Z_0 yields a twistor space, h_2 does not have critical points on I_2. Hence by Lemma 7.22, we can conclude that $\{\ell_1, \ell_2\} \neq \{x_0, x_0 + x_1\}$ and $\{\ell_1, \ell_2\} \neq \{x_1, x_0 - ax_1\}$. Namely, our investigation decreases the possibilities of small resolutions. We postpone further consequences until the next subsection.

7.5. **Consequences of the results in §7.2–7.4.** Before stating the results, we again recall our setup. Let B be a quartic surface defined by (3) and assume that Q and f satisfy the necessary conditions as in Proposition 2.6. Let $\Phi_0 : Z_0 \to \mathbf{CP}^3$ be the double covering branched along B. On $\mathbf{CP}^3$ there is a pencil of $U(1)$-invariant planes $\{H_\lambda\}$, where H_λ is

defined by $x_0 = \lambda x_1$ which is real iff $\lambda \in \mathbf{R} \cup \{\infty\}$. As explained in §7.3, an ordering $\ell_1, \ell_2, \ell_3, \ell_4$ of the four forms $\{x_0, x_1, x_0 + x_1, x_0 - ax_1\}$ determines a small resolution of p_∞ and $\overline{p}_\infty$ which are compound A_3-singularities of Z_0. $\mu_\infty : Z_0' \to Z_0$ denotes this small resolution. Then μ_∞ is only a partial resolution of Z_0; namely Z_0' still has a (unique and real) ordinary double point p_0 corresponding to the ordinary double point P_0 of B. For any small resolution $\nu : Z \to Z_0'$ of p_0, we put $\Phi := \Phi_0 \mu_\infty \nu$, and let $\{S_\lambda = \Phi^{-1}(H_\lambda)\}$ be (the real part of) a pencil of $U(1)$-invariant divisors on Z, where we put $S_\lambda = \Phi^{-1}(H_\lambda)$ as before.

We start with the following proposition, which uniquely determines the type of real touching conics which can be the images of twistor lines contained in $U(1)$-invariant fundamental divisors.

Proposition 7.23. *Suppose that there is a small resolution $\nu : Z \to Z_0'$ of the real ordinary double point p_0 such that Z is a twistor space. Let L be a twistor line of Z contained in S_λ for some $\lambda \in \mathbf{R}$. Then $\Phi(L)$ is a real touching conic of: (i) special type if $\lambda \in I_1 \cup I_3$, (ii) orbit type if $\lambda \in I_2$, (iii) generic type if $\lambda \in I_4$ and if $\lambda \neq \lambda_0$.*

Note that by Proposition 3.2, $\Phi(L) \subset H_\lambda$ is a line if $\lambda = \lambda_0$.

Proof. (i) immediately follows from (ii) of Proposition 5.6 and (v) of Proposition 6.3. (ii) is just Corollary 7.6. Finally we show (iii). By (i) of Proposition 5.6 it suffices to show that if $\lambda \in I_4$, the image cannot be of orbit type. In view of Lemma 7.22, we have $h_2(I_2) = (0, \infty)$ and $h_2(I_4) = (0, \infty)$ for any of the cases (i)–(iv) of the lemma. (We have already seen that the case (v) can be eliminated.) This implies that the circles appeared in Lemma 7.19 sweep out $\Gamma_2 \backslash \{\Gamma_2 \cap \Gamma_1, \Gamma_2 \cap \Gamma_3\}$. Therefore, $L_\alpha^\pm \subset S_\lambda$ with $\lambda \in I_2$, and $L_\alpha^\pm \subset S_\lambda$ with $\lambda \in I_4$ cannot be the images of twistor lines at the same time. Therefore, if $\lambda \in I_4$ and if $\lambda \neq \lambda_0$, the images of twistor lines must be of generic type, as required. □

The following is the main result of this section. Recall that p_∞ is a compound A_3-singularity of Z_0, and there are $4! = 24$ choices of small resolutions of p_∞, each one corresponding to a choice of ℓ_1, ℓ_2, ℓ_3 (and ℓ_4) (see Section 7.3). Recall also that once a resolution of p_∞ is given, it naturally induces that of $\overline{p}_\infty$ by reality.

Proposition 7.24. *Among 24 ways of possible small resolutions of p_∞, 22 resolutions do not yield a twistor space. The remaining two resolutions are given by the following two choices of linear forms:*

$$(I) \quad \ell_1 = x_1, \;\; \ell_2 = x_0 + x_1, \;\; \ell_3 = x_0,$$

and

$$(II) \quad \ell_1 = x_0 - ax_1, \;\; \ell_2 = x_0, \;\; \ell_3 = x_0 + x_1.$$

Here we do not yet claim that the threefolds obtained by these two resolutions are actually twistor spaces.

Proof. By Proposition 7.23 (i), if $\lambda \in I_1$, the images of twistor lines in S_λ are real touching conics of special type. As in Section 7.3, there are two families $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ of real rational curves which are candidates of twistor lines in S_λ. As we have already remarked in Section 6, $L_\theta^+ \in \mathcal{L}_\lambda^+$ and $L_\theta^- \in \mathcal{L}_\lambda^-$ cannot be twistor lines simultaneously. Suppose first that (any of the) members of $\mathcal{L}_\lambda^+$ are twistor lines. Then by Proposition 7.15, the function h_1 does not have critical points on I_1. By Lemma 7.16, this implies that we have either

$$\ell_1 = x_1 \quad \text{or} \quad \ell_1 = x_0 + x_1. \tag{53}$$

On the other hand, by Corollary 7.18 (i), under our assumption, members of $\mathcal{L}_\lambda^-$ are twistor lines for $\lambda \in I_3$. Therefore again by Proposition 7.15, h_3 does not have critical points on I_3. Then by Lemma 7.17, the cases (i) and (iii) of the lemma are eliminated and we have

either

$$\{\ell_1, \ell_2, \ell_3\} = \{x_1, x_0 + x_1, x_0 - ax_1\} \quad \text{or} \quad \{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, x_0 + x_1\}. \tag{54}$$

(Note again that we do not specify the order.)

Next we consider twistor lines in S_λ for $\lambda \in I_2$. By Proposition 7.23 (ii) the images are real touching conics of orbit type. Then Proposition 7.21 implies that h_2 has no critical point on I_2. Hence by Lemma 7.22, we have either

$$\{\ell_1, \ell_2\} = \{x_0, x_1\} \text{ or } \{x_0 + x_1, x_0 - ax_1\} \text{ or } \{x_1, x_0 + x_1\} \text{ or } \{x_0, x_0 - ax_1\}. \tag{55}$$

Now we note other restrictions: namely, when λ increases to pass from I_1 to I_2, twistor lines in S_λ must vary continuously, so that we have

$$\lim_{\lambda \uparrow -1} h_1(\lambda) = \left(\lim_{\lambda \downarrow -1} h_2(\lambda) \right)^{-1}. \tag{56}$$

(Here the inverse of the right hand side is a consequence of the fact that $\Gamma_1 \cap \Gamma_2 = \{u = 0\} = \{v = \infty\}$.) Similarly, moving λ from I_2 to I_3, we have

$$\lim_{\lambda \uparrow 0} h_2(\lambda) = \left(\lim_{\lambda \downarrow 0} h_3(\lambda) \right)^{-1}. \tag{57}$$

Take $\ell_1 = x_1$ for the first example. Then by Lemma 7.16 (i) we have $h_1(-1) = 0$. Hence it follows from (56) that $\lim_{\lambda \downarrow -1} h_2(\lambda) = \infty$. Then the cases (i) and (iv) of Lemma 7.22 fail and we have $\{\ell_1, \ell_2\} = \{x_0 + x_1, x_0 - ax_1\}$ ((ii)) or $\{\ell_1, \ell_2\} = \{x_1, x_0 + x_1\}$ ((iii)). The former clearly fails and we get $\ell_2 = x_0 + x_1$. This appears in (55). Then we have from Lemma 7.22 (iii) that $h_2(0) = 0$. Hence by (57), we have $\lim_{\lambda \downarrow 0} h_3(\lambda) = \infty$. It then follows from Lemma 7.17 that $\ell_3 = x_0$. Thus we get $\ell_1 = x_1, \ell_2 = x_0 + x_1, \ell_3 = x_0$.

Next take $\ell_1 = x_0 + x_1$. Then we have $\lim_{\lambda \uparrow -1} h_1(\lambda) = +\infty$ (Lemma 7.16 (iii)), so that $\lim_{\lambda \downarrow -1} h_2(\lambda) = 0$. Then looking (i)–(iv) of Lemma 7.22, this possibility fails. Namely, we have $l_1 \neq x_0 + x_1$. Thus we can conclude that if $L_\theta^+ \in \mathcal{L}_\lambda^+$ is supposed to be a twistor line over I_1, we have to choose $\ell_1 = x_1, \ell_2 = x_0 + x_1$ and $\ell_3 = x_0$. This is the item (I) of the theorem.

Next suppose that $L_\theta^- \in \mathcal{L}_\lambda^-$ is a twistor line over I_1 and repeat similar argument above. By Proposition 7.15, h_3 has no critical point on I_1. It then follows from Lemma 7.17 that either $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_0 + x_1, x_0 - ax_1\}$ ((i)) or $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, x_0 - ax_1\}$ ((iii)) holds. On the other hand, (55) is valid also in this case. Further we have as before

$$\lim_{\lambda \uparrow -1} h_3(\lambda) = \left(\lim_{\lambda \downarrow -1} h_2(\lambda) \right)^{-1} \quad \text{and} \quad \lim_{\lambda \uparrow 0} h_2(\lambda) = \left(\lim_{\lambda \downarrow 0} h_1(\lambda) \right)^{-1}. \tag{58}$$

If $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_0+x_1, x_0-ax_1\}$, then $\lim_{\lambda \uparrow -1} h_3(\lambda) = +\infty$ (Lemma 7.17 (i)), so that we have $\lim_{\lambda \downarrow -1} h_2(\lambda) = 0$ by (58). Hence by Lemma 7.22 we have $\{\ell_1, \ell_2\} = \{x_0, x_0 - ax_1\}$, which implies $\lim_{\lambda \uparrow 0} h_2(\lambda) = \infty$ ((iv) of Lemma 7.22) and $\ell_3 = x_0 + x_1$. Hence $\lim_{\lambda \downarrow 0} h_1(\lambda) = 0$ by (58). It follows from Lemma 7.16 that $\ell_1 = x_0 - ax_1$, which means $\ell_2 = x_0$. (iv) of Lemma 7.16 says that h_1 has no critical point on I_3, which is consistent with the fact that L_θ^+ is a twistor line over I_3.

If $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, x_0 - ax_1\}$, $h_3(-1) = 0$ (Lemma 7.17 (iii)), so that we have $\lim_{\lambda \downarrow -1} h_2(\lambda) = \infty$ by (58). Therefore we get the two possibilities (ii) and (iii) of Lemma 7.22, but both contain $x_0 + x_1$ which is not compatible with our choice of $\{\ell_1, \ell_2, \ell_3\}$. Thus we have $\{\ell_1, \ell_2, \ell_3\} \neq \{x_0, x_1, x_0 - ax_1\}$. This implies that if $L_\theta^- \in \mathcal{L}_\lambda^-$ is a twistor line for $\lambda \in I_1$, then $\ell_1 = x_0 - ax_1, \ell_2 = x_0$, and $\ell_3 = x_0 + x_1$. This is the case (II) of the theorem, and we have completed the proof. □

Next we summarize which irreducible component of the inverse image of touching conics have to be chosen as candidates of twistor lines. Note that for touching conic of special type, we have made in Section 7.3 distinction of the two components L_θ^+ and L_θ^- by the

property that L_θ^+ intersects Γ_1 and $\overline{\Gamma}_1$, and L_θ^- intersects Γ_3 and $\overline{\Gamma}_3$ (Lemma 7.14). For touching conics of generic type, we have not made distinction of L_θ^+ and L_θ^- so far. To make a distinction, we write $\Phi^{-1}(l_\infty) - \Gamma - \overline{\Gamma} = \Xi + \overline{\Xi}$, and Φ gives isomorphism of $\Xi \simeq l_\infty$ and $\overline{\Xi} \simeq l_\infty$ (cf. Proposition 2.10). We can suppose that Ξ intersects Γ_1 (and hence $\overline{\Gamma}_3$) as in Figure 8. Then $\overline{\Xi}$ intersects Γ_3 (and $\overline{\Gamma}_1$). Recall that C_θ intersects l_∞ at two points indicated by $e^{i\theta}h_0(\lambda)$ and $e^{-i\theta}h_0^{-1}(\lambda)$ (Lemma 7.2 and its proof). Thus L_θ^+ intersects Ξ at a point indicated by $e^{i\theta}h_0$ or $e^{-i\theta}h_0^{-1}$. Now we make a distinction of L_θ^+ and L_θ^- by declaring that $L_\theta^+ \cap \Xi$ is indicated by $e^{i\theta}h_0$ for $\lambda < \lambda_0$, and $e^{-i\theta}h_0^{-1}$ for $\lambda > \lambda_0$. It then follows that $L_\theta^- \cap \Xi$ is indicated by $e^{-i\theta}h_0^{-1}$ for $\lambda < \lambda_0$ and $e^{i\theta}h_0$ for $\lambda > \lambda_0$. The following result is essentially showed in the proof of Proposition 7.24.

Proposition 7.25. *(i) If we take the small resolution determined by (I) of Proposition 7.24, members of $\mathcal{L}_\lambda^+$ must be chosen as twistor lines on S_λ for $\lambda \in I_1$, both members L_α^+ and L_α^- must be chosen for $\alpha \in I_2$, members of $\mathcal{L}_\lambda^-$ must be chosen for $\lambda \in I_3$, and $\mathcal{L}_\lambda^+$ must be chosen for $\lambda \in I_4^- \cup I_4^+$. (ii) If we take the small resolution determined by (II) of Proposition 7.24, members of $\mathcal{L}_\lambda^-$ must be chosen as twistor lines on S_λ for $\lambda \in I_1$, both members L_α^+ and L_α^- must be chosen for $\lambda \in I_2$, members of $\mathcal{L}_\lambda^+$ must be chosen for $\lambda \in I_3$, and $\mathcal{L}_\lambda^-$ must be chosen for $\lambda \in I_4^- \cup I_4^+$.*

Proof. We only verify (i) because (ii) can be seen by parallel argument. The claims for $\lambda \in I_1 \cup I_3$ are already seen in the proof of Proposition 7.24. (ii) immediately follows form Proposition 6.3 (and its subsequent comment) and Corollary 7.6. It remains to see the claim for $\lambda \in I_4^\pm$. Over I_3, members of $\mathcal{L}_\lambda^-$ must be chosen as above. Further, we have $h_3(a) = 0$ by Lemma 7.17 (iv). This implies that as λ goes to a, the intersection circle $\cup\{\Gamma_3 \cap L_\theta^- \mid L_\theta^- \in \mathcal{L}_\lambda^-\}$ shrinks to be the point $\Gamma_3 \cap \overline{\Xi}$. Therefore the intersection circle of twistor lines in S_λ and $\overline{\Xi}$ also has to shrink to be $\Gamma_3 \cap \overline{\Xi}$. On $\overline{\Xi}$, one can use $x_2 = y_2/y_3$ as an affine coordinate whose center is the intersection point with Γ_3. By the rule explained just before the proposition, the radius (with respect to x_2) of intersection circle of $L_\theta^+ \cap \overline{\Xi}$ and $L_\theta^- \cap \overline{\Xi}$ are respectively indicated by h_0^{-1} and h_0. We have $\lim_{\lambda\downarrow a} h_0(\lambda) = +\infty$ and $\lim_{\lambda\downarrow a} h_0^{-1}(\lambda) = 0$ (cf. Figure 7). Hence we conclude that $L_\theta^+ \cap \overline{\Xi}$ must be chosen for $\lambda \in I_4^-$. Similar argument shows that for $\lambda \in I_4^-$, $L_\theta^+ \cap \overline{\Xi}$ must still be chosen. □

Differences of the two resolutions (I) and (II) of Proposition 7.24 can be displayed as in Figure 11. Briefly speaking, exchanging the two resolutions reverses the direction of the moving of the intersection circles as λ increases.

7.6. **Connectedness of the families of real lines obtained in §7.5.** In the last subsection we have determined the families of twistor lines contained in $S_\lambda = \Phi^{-1}(H_\lambda)$, where λ is in the intervals I_j, $1 \le j \le 4$. Namely, we have detected two S^1-families of real lines on S_λ for $\lambda \in \cup_{j=1}^4 I_j$, $\lambda \neq \lambda_0$. It is obvious from our explicit description that, for each $1 \le j \le 4$, the two S^1-families respectively forms a connected 2-dimensional family of real lines whose parameter space is a cylinder $I_j \times S^1$. As stated in Proposition 7.23, these families have different description depending on $1 \le j \le 4$. In this subsection, we show that, any member of these families convergent to the inverse image of the four tropes of B, when the plane moves to the four planes on which the tropes lie. This implies that the parameter spaces $I_j \times S^1$ are joined (connected) by adding four points corresponding to the tropes. This connected result will be needed in our proof of main theorem.

Recall that the branch quartic surface B is defined by

$$(y_2y_3 + Q(y_0, y_1))^2 - y_0y_1(y_0 + y_1)(y_0 - ay_1) = 0.$$

Let T_{-1}, T_0, T_a and T_∞ be the four tropes which are intersections of B with the planes $y_0 = -y_1, y_0 = 0, y_0 = ay_1$ and $y_1 = 0$ respectively. The inverse image of the tropes in Z_0

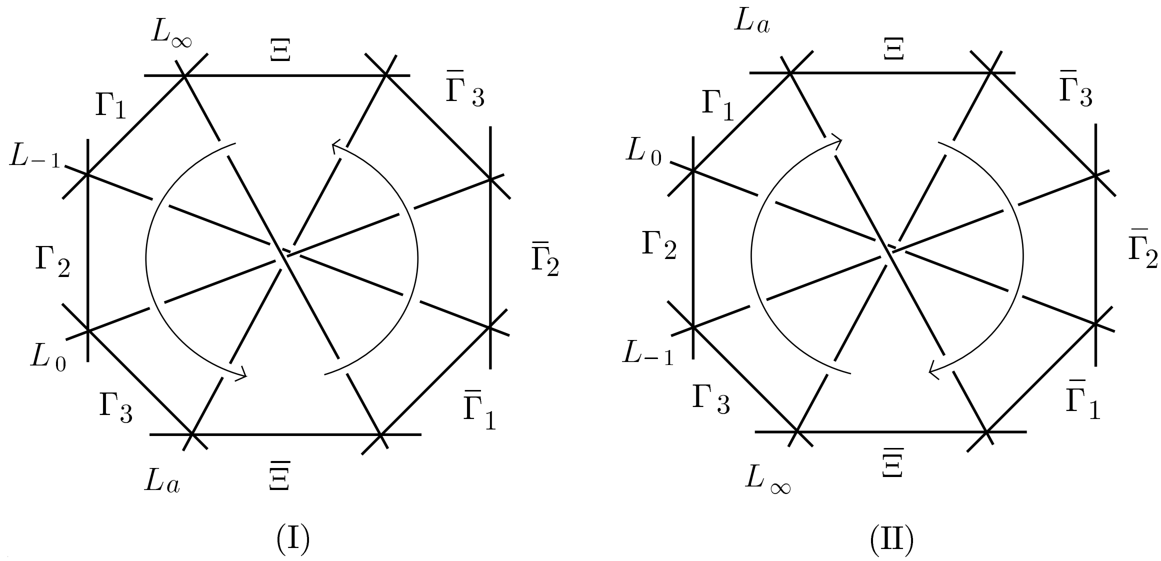

FIGURE 11. The intersection circles rotate as λ increases.

is of course a smooth rational curves which are mapped biholomorphically onto the tropes. These four curves on Z_0 define, as the strict transforms, smooth rational curves on Z_0' for any small resolution $Z_0' \to Z_0$ of p_∞ and $\overline{p}_\infty$. We denote these curves in Z_0' by L_{-1}, L_0, L_a and L_∞ respectively. These are $\mathbf{C}^*$-invariant real rational curves naturally biholomorphic to the tropes. By Lemma 7.10, the normal bundles in Z_0' are isomorphic to $O(1)^{\oplus 2}$ by Lemma 7.10. (In the lemma these were written L_{ℓ_i}.)

In the following we investigate limits of real lines on S_λ when λ approaches the endpoints of I_j. We begin with the case of $\lambda \in I_1 \cup I_3$. In this case we know by Proposition 7.23 that the images of twistor lines on S_λ is a touching conic of special type. Moreover, Proposition 7.25 shows that if we choose the small resolution determined by (I) of Proposition 7.24, then L_θ^+ must be selected as twistor lines on S_λ for $\lambda \in I_1$, whereas L_θ^- must be selected for $\lambda \in I_3$. Similarly, if we choose the small resolution determined by (II) of Proposition 7.24, then L_θ^- must be selected as twistor lines on S_λ with $\lambda \in I_1$, whereas L_θ^+ must be selected for $\lambda \in I_3$. The next two propositions in particular imply that when the real plane H_λ moves to the endpoints of I_1 and I_3, these real lines convergent to the real lines L_{λ_i}, $\lambda_i = -1, 0, a$ or ∞.

Proposition 7.26. *Assume $\lambda \in I_1 \cup I_3$ and let $L_\theta^\pm$ be the real lines in S_λ which are the irreducible components of the inverse image of real touching conic $C_\theta \subset H_\lambda$ of special type as in Section 7.3. If we choose the small resolution (I) of Proposition 7.24, we have the following:*

$$\begin{cases} \lim\limits_{\lambda\downarrow-\infty} L_\theta^+ = L_\infty, & \lim\limits_{\lambda\downarrow-\infty} L_\theta^- = L_\infty + \Gamma_1 + \Gamma_2 + \overline{\Gamma}_1 + \overline{\Gamma}_2, \\ \lim\limits_{\lambda\uparrow-1} L_\theta^+ = L_{-1}, & \lim\limits_{\lambda\uparrow-1} L_\theta^- = L_{-1} + \Gamma_2 + \overline{\Gamma}_2, \end{cases}$$

and

$$\begin{cases} \lim\limits_{\lambda\downarrow 0} L_\theta^+ = L_0 + \Gamma_2 + \overline{\Gamma}_2, & \lim\limits_{\lambda\downarrow 0} L_\theta^- = L_0, \\ \lim\limits_{\lambda\uparrow a} L_\theta^+ = L_a + \Gamma_2 + \Gamma_3 + \overline{\Gamma}_2 + \overline{\Gamma}_3, & \lim\limits_{\lambda\uparrow a} L_\theta^- = L_a. \end{cases}$$

Proof. It is immediate to verify from the explicit equation of touching conic of C_θ obtained in Proposition 5.4 that $\lim_{\lambda\downarrow-\infty} C_\theta = T_\infty$. Since $\Phi^{-1}(C_\theta) = L_\theta^+ + L_\theta^-$ and $\Phi^{-1}(T_\infty) = L_\infty + \Gamma + \overline{\Gamma}$, we obtain

$$\lim_{\lambda\downarrow-\infty} (L_\theta^+ + L_\theta^-) = L_\infty + \Gamma + \overline{\Gamma}.$$

Moreover, both $\lim_{\lambda\to-\infty} L_\theta^+$ and $\lim_{\lambda\to-\infty} L_\theta^-$ contain L_∞ as their irreducible components, since L_θ^+ and L_θ^- are mapped biholomorphically onto C_θ. First we consider $\lim L_\theta^+$ as $\lambda \downarrow -\infty$. By Lemma 7.14, L_θ^+ intersects Γ_1 at a unique point

$$u = -2iBe^{-i\theta} \cdot x_1/\ell_1 = ie^{-i\theta}h_1,$$

where u is the affine coordinate on Γ_1 introduced in the beginning of Section 7.3, and $h_1 = h_1(\lambda)$ is a function introduced before Proposition 7.15. Now since we are choosing the small resolution determined by (I) of Proposition 7.24, we have $\ell_1 = x_1$ and the graph of h_1^2 looks like (i) of Figure 9. (See Lemma 7.16 (i) for precisely.) In particular, we have $\lim_{\lambda\downarrow-\infty} h_1(\lambda) = \infty$. Because L_∞ is as in Figure 11 in this case, this implies that the intersection point $L_\theta^+ \cap \Gamma_1$ goes to the point $L_\infty \cap \Gamma_1$. Hence $\lim_{\lambda\downarrow-\infty} L_\theta^+$ does not contain any components of Γ and $\overline{\Gamma}$ and we get $\lim_{\lambda\downarrow-\infty} L_\theta^+ = L_\infty$. $\lim_{\lambda\uparrow-1} L_\theta^+ = L_{-1}$ can be proved in a similar way. Next we look into the limit of L_θ^- as λ approaches $-\infty$ and -1. We know by Lemma 7.14 that L_θ^- does not intersect $\Gamma_1 \cup \Gamma_2$ and that $L_\theta^- \cap \Gamma_3$ is a unique point satisfying

$$w = -\frac{ie^{i\theta} f}{2B} \cdot \frac{x_1^3}{\ell_1\ell_2\ell_3} = -ie^{i\theta}h_3,$$

where w is the affine coordinate on Γ_3 introduced in Section 7.3, and h_3 is a function introduced before Proposition 7.15. Moreover, by Lemma 7.17 (iv) (plus some easy calculations), we have $\lim_{\lambda\downarrow-\infty} h_3(\lambda) = \infty$ and $\lim_{\lambda\uparrow-1} h_3(\lambda) = \infty$. This implies that the point $L_\theta^- \cap \Gamma_3$ approaches $L_0 \cap \Gamma_3$ when $\lambda \downarrow -\infty$ and $\lambda \uparrow -1$. On the other hand, as is already mentioned, $\lim_{\lambda\downarrow-\infty} L_\theta^-$ contains L_∞ as its irreducible components. Then since $\lim_{\lambda\downarrow-\infty} L_\theta^-$ must be connected, it must contain Γ_1 and Γ_2 (see Figure 11). By reality, we conclude $\lim_{\lambda\downarrow-\infty} L_\theta^- = L_\infty + \Gamma_1 + \Gamma_2 + \overline{\Gamma}_1 + \overline{\Gamma}_2$. Similarly, $\lim_{\lambda\uparrow-1} L_\theta^-$ must contain Γ_2 and $\overline{\Gamma}_2$ and we get $\lim_{\lambda\uparrow-1} L_\theta^- = L_{-1} + \Gamma_2 + \overline{\Gamma}_2$, as claimed.

For other limits $\lambda \downarrow 0$ and $\lambda \uparrow a$, we can prove the claim of the proposition in a similar way by using Lemmas 7.16 and 7.17. We omit the detail. □

For another small resolutions, we have the following proposition. We omit the proof since it is completely parallel to the above case.

Proposition 7.27. *In Proposition 7.26, if we choose the small resolution (II) of Proposition 7.24 instead, we have the following:*

$$\begin{cases} \lim\limits_{\lambda\downarrow-\infty} L_\theta^+ = L_\infty + \Gamma_2 + \Gamma_3 + \overline{\Gamma}_2 + \overline{\Gamma}_3, & \lim\limits_{\lambda\downarrow-\infty} L_\theta^- = L_\infty, \\ \lim\limits_{\lambda\uparrow-1} L_\theta^+ = L_{-1} + \Gamma_2 + \overline{\Gamma}_2, & \lim\limits_{\lambda\uparrow-1} L_\theta^- = L_{-1}, \end{cases}$$

and

$$\begin{cases} \lim\limits_{\lambda\downarrow 0} L_\theta^+ = L_0 & \lim\limits_{\lambda\downarrow 0} L_\theta^- = L_0 + \Gamma_2 + \overline{\Gamma}_2,, \\ \lim\limits_{\lambda\uparrow a} L_\theta^+ = L_a, & \lim\limits_{\lambda\uparrow a} L_\theta^- = L_a + \Gamma_1 + \Gamma_2 + \overline{\Gamma}_1 + \overline{\Gamma}_2. \end{cases}$$

Next we investigate the limit of twistor lines on S_λ when λ approaches the endpoints of I_2. In this case we know by Proposition 7.23 that the image of twistor line is a touching conic of orbit type denoted by $C_\alpha \subset H_\lambda$, and that by Proposition 6.3 there are two irreducible components L_α^+ and L_α^- of $\Phi^{-1}(C_\alpha)$ which are biholomorphic to C_α by Φ. As remarked at the end of Section 6, both L_α^+ and L_α^- must be twistor lines (because they are members of the same pencil on S_λ).

Proposition 7.28. *Assume $\lambda \in I_2$ and let $L_\alpha^\pm$ be the real lines in S_λ which are irreducible components of the inverse image of real touching conic $C_\alpha \subset H_\lambda$ of orbit type as in Section 7.4. Then regardless of which small resolution (I) or (II) of Proposition 7.24 we take, we*

have the following

$$\lim_{\lambda\downarrow -1} L_\theta^+ = \lim_{\lambda\downarrow -1} L_\theta^- = L_{-1}, \quad \lim_{\lambda\uparrow 0} L_\theta^+ = \lim_{\lambda\uparrow 0} L_\theta^- = L_0.$$

Proof. This can be proved by the same argument as in Proposition 7.26, if we replace Proposition 5.4 by Proposition 5.5, Lemma 7.14 by Lemma 7.19, and Lemmas 7.16 and 7.17 by Lemma 7.22. □

Finally we study the case $\lambda \in I_4$. Again by Proposition 7.23 the image of twistor lines are real touching conics of generic type. We have $\Phi^{-1}(C_\theta) = L_\theta^+ + L_\theta^-$. Recall that in Section 7.5 (the explanation before Proposition 7.25) we have made distinction of L_θ^+ and L_θ^- by considering the intersection with Ξ and $\overline{\Xi}$.

Proposition 7.29. *Assume $\lambda \in I_4^- \cup I_4^+$ and let $L_\theta^\pm$ be as above. Then the following (I') and (II') respectively hold depending on which small resolution (I) and (II) of Proposition 7.24 we take:*

$$(I')\cdots\begin{cases} \lim\limits_{\lambda\downarrow a} L_\theta^+ = L_a, & \lim\limits_{\lambda\downarrow a} L_\theta^- = L_a + \Gamma + \overline{\Gamma}, \\ \lim\limits_{\lambda\uparrow\infty} L_\theta^+ = L_\infty, & \lim\limits_{\lambda\uparrow\infty} L_\theta^- = L_\infty + \Gamma + \overline{\Gamma}. \end{cases}$$

$$(II')\cdots\begin{cases} \lim\limits_{\lambda\downarrow a} L_\theta^+ = L_a + \Gamma + \overline{\Gamma}, & \lim\limits_{\lambda\downarrow a} L_\theta^- = L_a, \\ \lim\limits_{\lambda\uparrow\infty} L_\theta^+ = L_\infty + \Gamma + \overline{\Gamma}, & \lim\limits_{\lambda\uparrow\infty} L_\theta^- = L_\infty. \end{cases}$$

Proof. It is immediate to verify from the explicit equation of touching conic of generic type obtained in Proposition 5.2 that $\lim_{\lambda\downarrow a} C_\theta = T_a$ and $\lim_{\lambda\uparrow\infty} C_\theta = T_\infty$. As $\Phi^{-1}(C_\theta) = L_\theta^+ + L_\theta^-$ and $\Phi^{-1}(T_a) = L_a + \Gamma + \overline{\Gamma}$ and $\Phi^{-1}(T_\infty) = L_\infty + \Gamma + \overline{\Gamma}$, we obtain

$$\lim_{\lambda\downarrow a}(L_\theta^+ + L_\theta^-) = L_a + \Gamma + \overline{\Gamma}, \quad \lim_{\lambda\uparrow\infty}(L_\theta^+ + L_\theta^-) = L_\infty + \Gamma + \overline{\Gamma}.$$

By Lemma 7.4 (plus easy calculation; see Figure 7), we have $\lim_{\lambda\downarrow a} h_0(\lambda) = \infty$ and $\lim_{\lambda\uparrow\infty} h_0^{-1}(\lambda) = 0$. By the rule for making distinction of L_θ^+ and L_θ^-, these imply that $L_\theta^+ \cap \Xi$ goes to the point $\Xi \cap \overline{\Gamma}_3$ as $\lambda \downarrow a$, and goes to the point $\Xi \cap \Gamma_1$ as $\lambda \uparrow \infty$. On the other hand, we have $L_a \cap \Xi = \overline{\Gamma}_3 \cap \Xi$ and $L_\infty \cap \Xi = \Gamma_1 \cap \Xi$ for the small resolution determined by (I) of Proposition 7.24 (Figure 11). Namely as $\lambda \downarrow a$ and $\lambda \uparrow \infty$ respectively, $\Xi \cap L_\theta^-$ approaches $\Xi \cap L_a$ and $\Xi \cap L_\infty$. Therefore we have $\lim_{\lambda\downarrow a} L_\theta^+ = L_a$ and $\lim_{\lambda\uparrow\infty} L_\theta^+ = L_\infty$ if we take the small resolution coming from (I) of Proposition 7.24.

Next we look into the limit of L_θ^- as $\lambda \downarrow a$ and $\lambda \uparrow \infty$ for type (I) resolution. It is immediate from Lemma 7.4 and Figure 7 again to get that the point $\Xi \cap L_\theta^-$ goes to the points $\Xi \cap L_\infty$ as $\lambda \downarrow a$ and goes to the points $\Xi \cap L_a$ as $\lambda \uparrow \infty$. Then since $\lim_{\lambda\downarrow a} L_\theta^-$ is connected and contains L_a as its irreducible component (and since moreover the limit does not contain Ξ), it follows that $\lim_{\lambda\downarrow a} L_\theta^-$ contains Γ. Then by reality $\lim_{\lambda\downarrow a} L_\theta^-$ also contains $\overline{\Gamma}$. Thus we get $\lim_{\lambda\downarrow a} L_\theta^- = L_a + \Gamma + \overline{\Gamma}$. Similarly we get $\lim_{\lambda\uparrow\infty} L_\theta^- = L_\infty + \Gamma + \overline{\Gamma}$.

Finally, we can immediately obtain the limit for the type (II) resolution, if we note that exchanging resolution of type (I) and type (II) interchanges L_a and L_∞ (Figure 11). □

Propositions 7.26–7.29 can be summarized as follows:

Proposition 7.30. *Let $Z_0' \to Z_0$ be the small resolution of the conjugate pair of singularities determined by (I) or (II) of Proposition 7.24, and consider the families of real lines on S_λ ($\lambda \in I_j$) for each $1 \le j \le 4$ determined in Proposition 7.25, (i) or (ii) respectively. Then if we move H_λ into the planes on which the tropes of B lie, the real lines on S_λ converge to the real line which is (an irreducible component of) the inverse image of the trope.*

8. Twistor lines whose images are lines and a resolution of the ODP

In Sections 3–7 we have intensively studied real lines in Z whose images are touching conics. But as Proposition 3.2 says, we cannot obtain arbitrary twistor lines by considering such real lines only. In the first half of this section we study real lines whose images become lines. These real lines are needed for compactifying the space of real lines obtained in Sections 3–7.

On the other hand we showed in the last section that among many (twenty-four) ways of possible small resolutions of the conjugate pair of singularities of Z_0, there are just two resolutions which can yield a twistor space (Proposition 7.24). In the second half of this section we show that once a small resolution of the unique ordinary double point of Z_0 is given, it uniquely determines which one of the above two resolutions have to be taken for the conjugate pair of singularities. We keep the notations and assumptions in Sections 3–7.

As showed in Proposition 3.2, if $L \subset Z$ is a real line intersecting Γ_0, then $\Phi(L)$ is a line going through P_0. The next proposition is its converse.

Proposition 8.1. *Let $l \subset \mathbf{CP}^3$ be any real line going through P_0. Then $\Phi^{-1}(l)$ has just two irreducible components, both of which are real, smooth and rational. One of the components is the exceptional curve Γ_0 and another component is mapped (2 to 1) onto l. Further, the normal bundle of the latter component in Z is isomorphic to $O(1)^{\oplus 2}$.*

Proof. First we note that if l is a real line, $B \cap l$ consists of just three points, one of which is P_0. This follows from the facts that, B is a quartic, $B \cap l$ is real, P_0 is the unique real point of B (Proposition 2.5), and P_0 is a double point. Therefore $\Phi_0^{-1}(l) \to l$ is two-to-one covering branched at three points. Let P and $\overline{P}$ be the two branch points other than P_0. Because l intersects B transversally at these two points, $\Phi_0^{-1}(P)$ and $\Phi_0^{-1}(\overline{P})$ are smooth points of $\Phi_0^{-1}(l)$. Further, since P_0 is an ordinary double point of B, $p_0 = \Phi_0^{-1}(P_0)$ is a node of $\Phi_0^{-1}(l)$. From these it follows that $\Phi^{-1}(l)$ has just two irreducible components, one of which is Γ_0. Let L be the irreducible component different from Γ_0. Then L is smooth and $L \to l$ is two-to-one covering whose branch points are P and $\overline{P}$. Therefore (by Hurwitz) L is a rational curve. L is real since $\Phi_0^{-1}(l)$ is real.

It remains to show that $N_{L/Z} \simeq O(1)^{\oplus 2}$. The idea is similar to Propositions 7.3, 7.15, and 7.21. We first show that $N_{L/Z} \simeq O(1)^{\oplus 2}$ or $N_{L/Z} \simeq O \oplus O(2)$. By Bertini, $H \cap B$ is smooth outside P_0 for general plane H containing l. Further, since $H \cap B$ is a quartic, $S := \Phi^{-1}(H)$ is a smooth rational surface with $c_1^2 = 2$. Moreover, $\Phi^{-1}(l)$ is an anticanonical curve of S so that we have $(\Gamma_0 + L)^2 = 2$ on S. Furthermore, it is readily seen that $\Gamma_0^2 = -2$ and $\Gamma_0 \cdot L = 2$ on S. Therefore we have $L^2 = 0$ on S. Then the argument in the proof of Proposition 7.3 implies $N_{L/Z} \simeq O(1)^{\oplus 2}$ or $N_{L/Z} \simeq O(2) \oplus O$. To show that the latter does not hold, we first see that $\Gamma_0/\langle\sigma\rangle$ is canonically identified with the projective space of real lines going through P_0. Concretely, for each real line $l \ni P_0$, we associate the intersection $\Gamma_0 \cap (\Phi^{-1}(l) - \Gamma_0)$ which is a conjugate pair of points. We show by explicit calculation that this correspondence, which we will denote by ψ, is actually an isomorphism. The problem being local, we use a local coordinate (w_1, w_2, w_3) (around P_0) defined in (4). Then in a neighborhood of $p_0 = \Phi_0^{-1}(P_0)$, Z_0 is given by the equation

$$z^2 + w_1^2 + w_2 w_3 = 0. \tag{59}$$

Small resolutions of the double point $p_0 \in Z_0$ are explicitly obtained by blowing-up along $\{z + iw_1 = w_3 = 0\}$ or $\{z + iw_1 = w_2 = 0\}$. In the former case, we can use $(z + iw_1 : w_3) = (-w_2 : z - iw_1)$ as a homogeneous coordinate on Γ_0, whereas in the latter case we can use $(z + iw_1 : w_2) = (-w_3 : z - iw_1)$ instead. We see only in the former case, since the calculation is identical. Let $(w_1 : w_2 : w_3)$ be a real line through P_0. Namely, we assume $w_1 \in \mathbf{R}$, $\overline{w}_2 = w_3$, and $w_1^2 + |w_2|^2 \neq 0$. Then by (59), we have $z = \pm i(w_1^2 + |w_2|^2)^{1/2}$. Hence

we get $(z+iw_1:w_3)=(i(w_1\pm(w_1^2+|w_2|^2)^{1/2}):\overline{w}_2)$. Namely, ψ is explicitly given by

$$\psi:(w_1:w_2:\overline{w}_2)\longmapsto\left(i\left(w_1\pm\sqrt{w_1^2+|w_2|^2}\right):\overline{w}_2\right). \tag{60}$$

(Note that the image of (60) is considered as a point of $\Gamma_0/\langle\sigma\rangle$.) First suppose $w_1\neq 0$. It is readily seen that we can suppose $w_1=1$. Then in (60) the image becomes $(i(1\pm(1+|w_2|^2)^{1/2}):\overline{w}_2)$. Taking the sign '+', (60) can be rewritten as

$$\psi:\mathbf{C}\ni w_2\mapsto\frac{-i\overline{w}_2}{1+\sqrt{1+|w_2|^2}}, \tag{61}$$

where we use (the second entry)/(the first entry) as an affine coordinate on Γ_0. The image of (61) is clearly contained in the unit disk $\{u\in\mathbf{C}\,|\,|u|<1\}$. We show that (61) give a diffeomorphism between $\mathbf{C}=\mathbf{R}^2$ and the unit disk. Putting $w_2=re^{i\theta}$, (61) is rewritten as

$$\psi:re^{i\theta}\longmapsto\frac{-ire^{-i\theta}}{1+\sqrt{1+r^2}}. \tag{62}$$

It is elementary to show that $k(r):=r/(1+\sqrt{1+r^2})$ is differentiable on $\{r>0\}$ and its derivative is always positive, and that $\lim_{r\uparrow\infty}k(r)=1$ and $\lim_{r\downarrow 0}k(r)=0$ hold. Hence k gives a bijection between $\{r\geq 0\}$ and $\{0\leq s<1\}$. It follows that (61) gives a bijection between $\mathbf{C}$ and the unit disk. Moreover, the positivity of k' implies that (62) is a diffeomorphism on $\mathbf{C}^*$. For $w_2=0$, it can be easily checked that $(\partial w_2/\partial\overline{w}_2)(0)\neq 0$. Therefore (61) is a diffeomorphism on $\mathbf{C}$.

Next consider the case $w_1=0$. Then we have $w_2\neq 0$, and the image becomes $(\pm i|w_2|:\overline{w}_2)=(1:\pm i\overline{w}_2/|w_2|)$. From this, it easily follows that (60) gives a diffeomorphism between the two subsets $\{(0:1:w)\,|\,w\in U(1)\}$ and $\{(1:u)\in\Gamma_0\,|\,u\in U(1)\}/\langle\sigma\rangle$. Moreover on $\mathbf{RP}^2\backslash\mathbf{R}^2$ we can use $(1/r,\theta)$ as a local coordinate on $\mathbf{RP}^2$. Then we can readily show that $(d/ds)(k(1/s))|_{s=0}\neq 0$. This implies that ψ is diffeomorphic also on a neighborhood of $\mathbf{RP}^2\backslash\mathbf{R}^2$. Note that the bijectivity of ψ implies that if $l\neq l'$, then the corresponding rational curves in Z are disjoint.

Next take any real plane H containing l. On H there is a one-dimensional family of lines through P_0. Taking the inverse image, we obtain a one-dimensional holomorphic family $\mathcal{L}_H$ of rational curves in Z, containing L as a real member. Any real member of $\mathcal{L}_H$ defines a conjugate pair of points as the intersection with Γ_0. Consequently, real members of $\mathcal{L}_H$ determine a real circle $\mathcal{C}_H$ in Γ_0. If s denotes the section of $N=N_{L/Z}$ associated to $\mathcal{L}_H$ (viewed as a deformation of L in Z), then $\mathrm{Re}s(z)$ is non-zero by the diffeomorphicity of ψ, and is represented by a tangent vector of $\mathcal{C}_H$ at z, where we put $\{z,\overline{z}\}=\Gamma_0\cap L$.

Let $\{v_1,v_2\}$ be any oriented orthogonal basis of $T_z\Gamma_0$, where we take the complex orientation and orthogonality. Then since we have the isomorphism ψ, there is a unique real plane H_i $(i=1,2)$ containing l such that v_i is tangent to $\mathcal{C}_{H_i}$. Let s (resp. t) be the global section of $N=N_{L/Z}$ associated to $\mathcal{L}_{H_1}$ (resp. $\mathcal{L}_{H_2}$). We now claim that $as+bt$ does not vanish at z and $\overline{z}$ simultaneously, unless $(a,b)=(0,0)$. Putting $a=a_1+ia_2$ and $b=b_1+ib_2$, we readily have

$$\mathrm{Re}(as+bt)=(a_1\mathrm{Re}s-b_2\mathrm{Im}t)+(b_1\mathrm{Re}t-a_2\mathrm{Im}s). \tag{63}$$

Since $(\mathrm{Re}s)(z)$ and $(\mathrm{Re}s)(\overline{z})$ (resp. $(\mathrm{Re}t)(z)$ and $(\mathrm{Re}t)(\overline{z})$) are represented by tangent vectors of $\mathcal{C}_{H_1}$ (resp. $\mathcal{C}_{H_2}$), our choice of H_1 and H_2 implies that $(\mathrm{Re}s)(z)$ is parallel to $(\mathrm{Im}t)(z)$ and $(\mathrm{Re}t)(z)$ is parallel to $(\mathrm{Im}s)(z)$. The same is true at $\overline{z}$. Hence by (63) if $\mathrm{Re}(as+bt)(z)=0$, then $a_1\mathrm{Re}s(z)=b_2\mathrm{Im}t(z)$ and $b_1\mathrm{Re}t(z)=a_2\mathrm{Im}s(z)$, and $\mathrm{Re}(as+bt)(\overline{z})=0$ implies similar equalities. Since $\mathrm{Re}s,\mathrm{Re}t,\mathrm{Im}s$ and $\mathrm{Im}t$ do not be zero at both of z and $\overline{z}$ as is already mentioned, $a_1=0$ iff $b_2=0$ and $a_2=0$ iff $b_1=0$. Therefore either $a_1b_2\neq 0$ or $b_1a_2\neq 0$ holds. Suppose $a_1b_2\neq 0$. Then we show that $a_1\mathrm{Re}s(z)=b_2\mathrm{Im}t(z)$ and $a_1\mathrm{Re}s(\overline{z})=b_2\mathrm{Im}t(\overline{z})$ cannot hold simultaneously: suppose that $a_1b_2>0$. Then $\mathrm{Re}s(z)$ and $\mathrm{Im}t(z)$ have the same

direction and it follows that $\{\mathrm{Re}t(z), \mathrm{Re}s(z)\,(= (b_2/a_1)\mathrm{Im}t(z))\}$ is an oriented basis of $T_z\Gamma_0$. On the other hand, we have $\mathrm{Re}s(\overline{z}) = \sigma_*(\mathrm{Re}s(z))$ and $\mathrm{Re}t(\overline{z}) = \sigma_*(\mathrm{Re}t(z))$. Therefore $\{\mathrm{Re}t(\overline{z}), \mathrm{Re}s(\overline{z})\}$ is an anti-oriented basis of $T_{\overline{z}}\Gamma_0$ because σ is orientation reversing. On the other hand, $a_1\mathrm{Re}s(\overline{z}) = b_2\mathrm{Im}t(\overline{z})$ and $a_1b_2 \neq 0$ imply that $\{\mathrm{Re}t(\overline{z}), \mathrm{Re}s(\overline{z})\}$ is an oriented basis of $T_{\overline{z}}\Gamma_0$. This is a contradiction. The case $b_1a_2 > 0$ is similar. Therefore $\mathrm{Re}(as + bt)$ cannot be zero at z and $\overline{z}$ simultaneously provided $a_1b_2 \neq 0$. If $b_1a_2 \neq 0$, then $b_1\mathrm{Re}t(z) = a_2\mathrm{Im}s(z)$ and $b_1\mathrm{Re}t(\overline{z}) = a_2\mathrm{Im}s(\overline{z})$ do not hold at the same time. Thus we have shown that $\mathrm{Re}(as + bt)$ cannot be zero at z and $\overline{z}$ simultaneously for any $(a, b) \neq (0, 0)$. Hence so does $as + bt$. Therefore we get $N \simeq O(1)^{\oplus 2}$ by Lemma 7.1 and complete a proof of Proposition 8.1. □

Thus in our complex manifold Z there actually exists a connected family of real lines. Obviously this family is $U(1)$-invariant, although general members are not $U(1)$-invariant:

Proposition 8.2. *Among this family of real lines in Z, just one member is $U(1)$-invariant. Further, the member is fixed by $U(1)$ pointwisely.*

Proof. Recall that in a neighborhood of p_0, Z_0 is defined by the equation $z^2 + w_1^2 + w_2w_3 = 0$ ((59)). It is immediate to see that the $U(1)$-action looks like $(w_1, w_2, w_3) \mapsto (w_1, tw_2, t^{-1}w_3)$ for $t \in U(1)$. Thus using homogeneous coordinates used in the last proof, the $U(1)$-action on Γ_0 is given by $(u : v) \mapsto (u : tv)$ or $(u : v) \mapsto (u : t^{-1}v)$, depending on the choice of a small resolution of p_0. Therefore only the real line corresponding to $[(1 : 0)](= [(0 : 1)]) \in \Gamma_0/\langle\sigma\rangle$ is $U(1)$-fixed. In view of (60) and (4), the equation of this line is explicitly given by $y_2 = y_3 = 0$, which is pointwisely $U(1)$-fixed by Proposition 2.1. Since $\Phi : Z \to \mathbf{CP}^3$ is $U(1)$-equivariant, it follows that the corresponding real line in Z is also pointwise fixed. □

Definition 8.3. We will call the real lines in Z obtained by Proposition 8.1 real lines *at infinity*. Namely, a real line is said to be at infinity if its image in $\mathbf{CP}^3$ (by Φ) is a line (necessarily going through P_0).

Because there are $\mathbf{RP}^2$'s worth of real lines in $\mathbf{CP}^3$ going through P_0, real lines at infinity are parametrized by $\mathbf{RP}^2$. In the rest of this section we will study deformations in Z of real lines at infinity.

For any real plane H going through P_0, we define a line bundle $\mathcal{L}_H$ over the smooth surface $S_H = \Phi^{-1}(H)$ (cf. Proposition 3.5) by

$$\mathcal{L}_H = \Phi^*O_H(1) - \Gamma_0.$$

Of course, we consider Γ_0 as a divisor on S_H. Clearly $|\mathcal{L}_H|$ is a real pencil on S_H and its real part is precisely the family of real lines at infinity lying on S_H. Since S_H is rational, $\mathcal{L}_H \to S_H$ uniquely extends to a line bundle $\mathcal{L}_{H'} \to S_{H'}$ for any real plane H' sufficiently near to H, and that $|\mathcal{L}_{H'}|$ is still a real pencil whose real members are smooth rational curves parametrized by S^1. Let us consider the particular case that the above planes H and H' contain l_∞. Namely $H = H_{\lambda_0}$ and $H' = H_\lambda \in \langle l_\infty\rangle^\sigma$ with $\lambda \in I_4^- \cup I_4^+$ in the notations of Section 7. As introduced in Section 7.2, there are two families

$$\mathcal{L}_\lambda^+ = \{L_\theta^+ \subset S_\lambda \,|\, e^{i\theta} \in U(1)\}, \;\; \mathcal{L}_\lambda^- = \{L_\theta^- \subset S_\lambda \,|\, e^{i\theta} \in U(1)\}$$

of real lines lying on $S_\lambda = S_{H_\lambda}$; L_θ^+ and L_θ^- are irreducible components of the inverse images of touching conics of generic type on H_λ. It is expected that either $\mathcal{L}_\lambda^+$ or $\mathcal{L}_\lambda^-$ is precisely the family obtained by deforming real lines at infinity. This is true, since the image of real line not at infinity is always a real touching conic (Proposition 3.2) and since touching conics of generic type are the only one which do not go through P_∞ and $\overline{P}_\infty$ on H_λ (Propositions 5.2, 5.4 and 5.5), and since being real line is preserved under small deformation. Now recall that the distinction of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ ($\lambda \in I_4^- \cup I_4^+$) made in Section 7.6. We have been assumed $\Xi \cap L_\theta^- = \{x_2 = e^{i\theta}h_0(\lambda)\}$ for $\lambda < \lambda_0$ and $\Xi \cap L_\theta^- = \{x_2 = e^{-i\theta}h_0^{-1}(\lambda)\}$ for $\lambda > \lambda_0$, where

$x_2 = y_2/y_3$ is used as a coordinate on $\Xi \simeq l_\infty$ and h_0 is a function on I_4 introduced in Section 7.2 that behaves as in Figure 7. Assume for instance that members of $\mathcal{L}_\lambda^+$ are obtained as a deformation of real lines at infinity for $\lambda < \lambda_0$. We claim that this assumption implies that the same is true for $\lambda > \lambda_0$. Suppose not. Then members of $\mathcal{L}_\lambda^-$ must be deformations of real lines at infinity. However, the behavior of h_0 displayed in Figure 7 shows that, for any $\lambda < \lambda_0$ and for any $L_\theta^+ \in \mathcal{L}_\theta^+$ there exists $\lambda' > \lambda_0$ such that $L_\theta^- \in \mathcal{L}_{\lambda'}^-$ intersects L_θ^+ on Ξ. Namely $L_\theta^+ \subset S_\lambda$ intersects $L_\theta^- \subset S_{\lambda'}$ (cf. Proposition 7.8). On the other hand, we already know that real lines at infinity has the right normal bundle (Proposition 8.1) and therefore Z has a structure of twistor space, at least on a neighborhood of real lines at infinity. In particular, $L_\theta^+ \subset S_\lambda$ and $L_\theta^- \subset S_{\lambda'}$ must be disjoint, because they are assumed to be obtained by deforming real lines at infinity. This is a contradiction and we obtain that members of $\mathcal{L}_\lambda^+$ is obtained as a deformation of real lines at infinity for $\lambda > \lambda_0$. By the same reason, if members of $\mathcal{L}_\lambda^-$ are obtained as deformation of real lines at infinity for $\lambda < \lambda_0$, then the same is true for $\mathcal{L}_\lambda^-$ for $\lambda > \lambda_0$. Thus we have proved that real lines at infinity 'connects' $\mathcal{L}_\lambda^+(\lambda < \lambda_0)$ and $\mathcal{L}_\lambda^+(\lambda > \lambda_0)$, or $\mathcal{L}_\lambda^-(\lambda < \lambda_0)$ and $\mathcal{L}_\lambda^-(\lambda > \lambda_0)$. It is obvious that just one of these two situations happens. The following proposition shows that which one happens depends on the choice of a small resolution of p_0.

Proposition 8.4. *Changing a small resolution of $p_0 \in Z_0$ switches which one of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ $(\lambda \in I_4)$ is obtained as deformation of real lines at infinity . Namely, which irreducible component of $\Phi^{-1}(C_\theta)$ (C_θ being a touching conic of generic type) is a deformation of real lines at infinity depends on the choice of small resolution of p_0.*

Proof. Recall that by Proposition 5.2 touching conic of generic type on H_λ is defined by

$$C_\theta : \quad 2(Q^2 - f)y_1^2 + \sqrt{f}e^{i\theta}y_2^2 + 2Qy_2y_3 + \sqrt{f}e^{-i\theta}y_3^2 = 0. \tag{64}$$

(We always need to keep in mind that C_θ depends on λ.) Also recall that H_λ contains P_0 precisely when $\lambda = \lambda_0$ which satisfies $(Q^2 - f)(\lambda_0) = 0$. Hence substituting $\lambda = \lambda_0$ into the above equation, we get $e^{i\theta}y_2^2 + 2y_2y_3 + e^{-i\theta}y_3^2 = 0$. This can be written as $(e^{\frac{\theta}{2}}y_2 + e^{-\frac{\theta}{2}}y_3)^2 = 0$. Let l_θ be the line on H_{λ_0} defined by $e^{\frac{\theta}{2}}y_2 + e^{-\frac{\theta}{2}}y_3 = 0$ (and $y_0 = \lambda_0 y_1$). We have seen that $\lim_{\lambda\to\lambda_0} C_\theta = l_\theta$. Taking the inverse image, we obtain

$$\lim_{\lambda\to\lambda_0} \Phi^{-1}(C_\theta) = L_\theta + \Gamma_0,$$

where L_θ denotes the irreducible component of $\Phi^{-1}(l_\theta)$ that are mapped surjectively onto l_θ. Since $\Phi^{-1}(C_\theta) = L_\theta^+ + L_\theta^-$, it follows that either

$$\lim_{\lambda\to\lambda_0} L_\theta^+ = L_\theta, \qquad \lim_{\lambda\to\lambda_0} L_\theta^- = L_\theta + \Gamma_0 \tag{65}$$

or

$$\lim_{\lambda\to\lambda_0} L_\theta^+ = L_\theta + \Gamma_0, \qquad \lim_{\lambda\to\lambda_0} L_\theta^- = L_\theta \tag{66}$$

holds. To prove the proposition it suffices to show the following:

($*$) Changing a choice of small resolution of p_0 interchanges which one of (65) and (66) happens.

By applying $U(1)$-action, it suffices to prove ($*$) for the case $\theta = 0$. We write $C_{\theta=0}$ and $L_{\theta=0}^\pm$ to mean the curves obtained by substituting $\theta = 0$ into C_θ and L_θ respectively. We note that two small resolutions of p_0 given in the proof of Proposition 8.1 are also obtained by first taking the blowing-up at p_0 (in the usual sense) and then blowing-down the exceptional divisor ($\simeq \mathbf{CP}^1 \times \mathbf{CP}^1$) in the two directions. Consider $\cup_\lambda L_{\theta=0}^+$ and $\cup_\lambda L_{\theta=0}^-$ which are surfaces in Z_0. (Remind that $L_\theta^\pm$ are contained in S_λ and depend on λ.) To prove

(*) we consider *the tangent cones* at p_0 of these surfaces. As in the proof of Proposition 2.2 putting $v_i = y_i/y_1$ and $(Q^2 - f)(v_0) = g(v_0)(v_0 - \lambda_0)^2$, we can use three functions

$$w_1 = \sqrt{g(v_0)} \cdot (v_0 - \lambda_0), w_2 = \sqrt{2Q(v_0, 1) + v_2 v_3} \cdot v_2, w_3 = \sqrt{2Q(v_0, 1) + v_2 v_3} \cdot v_3$$

as a local coordinate around P_0. Then rewriting (64), the equation of $C_{\theta=0} \subset H_\lambda$ in a neighborhood of P_0 becomes

$$w_1 = \sqrt{g}(\lambda - \lambda_0),$$
$$\sqrt{f}\left(w_2^2 + w_3^2\right) + 2\left\{(Q^2 - f)\sqrt{Q^2 + w_2 w_3} + Q\left(Q^2 - f + w_2 w_3\right)\right\} = 0.$$

Then since we have $Q^2 - f = w_1^2$ and since $Q(\lambda_0, 1) \neq 0$ and $f(\lambda_0) \neq 0$ (basically because we are assuming Condition (A) of Proposition 2.6), a surface $\cup_\lambda C_{\theta=0}$ is singular at P_0 and its tangent cone is

$$\sqrt{f}\left(w_2^2 + w_3^2\right) + 2Qw_1^2 + 2Q\left(w_1^2 + w_2 w_3\right) = 0,$$

where we consider Q and f as a function of w_1. But since $Q(\lambda_0, 1) = \sqrt{f(\lambda_0)}$ we get an equation of the tangent cone of $\cup_\lambda C_{\theta=0}$ to be

$$4w_1^2 + 2w_2 w_3 + w_2^2 + w_3^2 = 0.$$

Thus the tangent cone splits into a union of two planes $2w_1 = \pm i(w_2 + w_3)$. On the other hand, in a neighborhood of p_0, Z_0 was defined by $z^2 + w_1^2 + w_2 w_3 = 0$. Hence a equation of the tangent cone of $\cup_\lambda \Phi_0^{-1}(C_{\theta=0})$ at p_0 becomes

$$\{2z - (w_2 - w_3)\}\{2z + (w_2 - w_3)\} = 0, \;\; 2w_1 = \pm i(w_2 + w_3).$$

which is also a union of two planes. On the exceptional divisor of blowing-up at p_0, these equations define a reduced reducible curve of bidegree $(2, 2)$ constituting of four irreducible components, two of which are conjugate pair of curves of bidegree $(1, 0)$ and the other two are conjugate pair of curves of bidegree $(0, 1)$. Hence we get that the tangent cone of $\cup_\lambda \Phi_0^{-1}(C_{\theta=0})$ at p_0 defines, on the exceptional divisor ($\simeq \mathbf{CP}^1 \times \mathbf{CP}^1$), the reducible curve of bidegree $(2, 2)$. On the other hand, we know that each irreducible component of $\Phi_0^{-1}(C_\theta)$ is real (Proposition 6.1) (iv)). Moreover, easy calculations using local coordinate shows that real structure on the exceptional divisor acts as a product of complex conjugations on each factor and does not exchange the factors. Therefore, on the full-blowup, either

$$\lim_{\lambda \to \lambda_0} L^+_{\theta=0} = L_{\theta=0} + (2, 0), \;\; \lim_{\lambda \to \lambda_0} L^-_{\theta=0} = L_{\theta=0} + (0, 2)$$

or

$$\lim_{\lambda \to \lambda_0} L^+_{\theta=0} = L_{\theta=0} + (0, 2), \;\; \lim_{\lambda \to \lambda_0} L^-_{\theta=0} = L_{\theta=0} + (2, 0)$$

holds, where $(2, 0)$ (resp. $(0, 2)$) means a reducible curve of bidegree $(2, 0)$ (resp. $(0, 2)$) that is a conjugate pair of curves of bidegree $(1, 0)$ (resp. $(0, 1)$). If we blow down the exceptional divisor along the direction whose fibers are curves of bidegree $(2, 0)$, we get, on the blown-down space Z, that either

$$\lim_{\lambda \to \lambda_0} L^+_{\theta=0} = L_{\theta=0}, \;\; \lim_{\lambda \to \lambda_0} L^-_{\theta=0} = L_{\theta=0} + \Gamma_0$$

or

$$\lim_{\lambda \to \lambda_0} L^+_{\theta=0} = L_{\theta=0} + \Gamma_0, \;\; \lim_{\lambda \to \lambda_0} L^-_{\theta=0} = L_{\theta=0}$$

holds, depending on the above two cases respectively. Also we have an alternative conclusion for another blow-down of the exceptional divisor. This implies that just one of (65) and (66) happens, depending on the direction of blow-down of the exceptional divisor. This implies the desired conclusion. $\square$

Combined Proposition 7.30 with Proposition 8.4, we obtain the following proposition which will be needed in our proof of the main theorem.

Proposition 8.5. *For each of the two small resolutions $Z_0' \to Z_0$ determined in Proposition 7.24, there exists a unique small resolution $\nu : Z \to Z_0'$ of the ordinary double point satisfying the following properties:*

Let $\mu : Z \to Z_0$ be the composition of the two resolutions, and put $\Phi = \Phi_0\mu$. Then (i) Φ preserves the real structure, (ii) for each $H_\lambda \in \langle l_\infty \rangle^\sigma$, $\lambda \in I_1 \cup I_2 \cup I_3 \cup I_4$, there exists a family of real lines on $S_\lambda = \Phi^{-1}(H_\lambda)$ whose parameter space is S^1, thereby getting families of real lines parametrized by $I_j \times S^1$, $1 \le j \le 4$, (iii) for $\lambda = -1, 0, a, \infty$, the intersection of the two irreducible components of $\Phi^{-1}(H_\lambda)$ is a real line in Z, (iv) when λ approaches the endpoints of I_j, the real lines in (ii) converge to the real lines in (iii), (v) if $\lambda = \lambda_0$, the S^1-family of real lines on S_{λ_0} in (ii) are real lines at infinity, (vi) when $\lambda \in I_4^- \cup I_4^+$ approaches λ_0, the real lines on S_λ obtained in (ii) converge to the real lines at infinity.

Moreover, if we change the small resolution ((I) and (II) of Proposition 7.24), then the above resolution of the ordinary double point also changes.

The point of this proposition is (iv) and (v) which imply that the S^1-family of real lines on S_λ, $\lambda \in I_1 \cup I_2 \cup I_3 \cup I_4^- \cup I_4^+$ obtained in (ii) form a connected family, whose parameter space is a union of four spheres, where 'North pole' of each sphere is identified with 'South pole' of the next sphere.

9. Global construction of twistor lines and their disjointness

In Section 7.5 we have proved that there are only two (small) resolutions of the conjugate pair of singularities of Z_0 which can yield a twistor space (Proposition 7.24). In the previous section we have shown that once one of the above two resolutions is given, a resolution of the remaining singularity (the real ordinary double point) is automatically determined and that, then, real lines contained in S_λ, $\lambda \in \mathbf{R} \cup \{\infty\}$ naturally forms a connected family. Because the parameter space of this family is real two-dimensional, and because we already know that the normal bundles in Z of the members are $O(1)^{\oplus 2}$, the family is not a complete family. In this section we first show that members of this family can be extended in Z to give (real) four-dimensional family of real rational curves (Proposition 9.1). Next we show that these real rational curves are disjoint and further cover Z; namely they foliate Z (Proposition 9.3).

For these purposes, we briefly recall notations and results in Section 3. Let $(\mathbf{RP}^3)^\vee$ be the dual projective planes of real planes in $\mathbf{CP}^3$, and $\mathbf{RP}^2_\infty \subset (\mathbf{RP}^3)^\vee$ the set of real planes going through P_0. Then $\langle l_\infty \rangle^\sigma$, the set of real planes containing l_∞, becomes a line in $(\mathbf{RP}^3)^\vee$, intersecting $\mathbf{RP}^2_\infty$ transversally at a point (for which we have denoted H_{λ_0}). Clearly we have $(\mathbf{RP}^3)^\vee \backslash \mathbf{RP}^2_\infty = \mathbf{R}^3$. By Lemma 3.4, $\mathbf{R}^3 \backslash \langle l_\infty \rangle^\sigma$ is precisely the set of real planes intersecting B smoothly; namely we have $U^\sigma = \mathbf{R}^3 \backslash \langle l_\infty \rangle^\sigma (\simeq \mathbf{C}^* \times \mathbf{R})$. We have constructed a fibration $\mathcal{C}_{U^\sigma} \to U^\sigma$ whose fiber over $H \in U^\sigma$ is the set of touching conics of $H \cap B$. $\mathcal{C}_{U^\sigma} \to U^\sigma$ is a fiber bundle whose fiber is a union of 63 copies of $\mathbf{CP}^1$ (Proposition 3.11). Recalling that the inverse image of touching conics splits into two curves and taking the inverse image by the double cover $Z_0 \to \mathbf{CP}^3$, we get a fiber bundle $\tilde{\mathcal{C}}_{U^\sigma} \to U^\sigma$ whose fiber is a union of 128 copies of $\mathbf{CP}^1$. Each $\mathbf{CP}^1$ of the fibers of $\tilde{\mathcal{C}}_{U^\sigma} \to U^\sigma$ is a parameter space of pencils on S_H, whose image is a one-dimensional family of touching conics. The following proposition gives all twistor lines in Z:

Proposition 9.1. *For any real plane H different from the four planes on which the tropes of B lie, we can find a S^1-family of real smooth rational curves on $S_H = \Phi^{-1}(H)$ satisfying the following properties: (i) if $H \in \langle l_\infty \rangle^\sigma$, the S^1-family on S_H is just the family of real lines obtained in (ii) of Proposition 8.5, (ii) if $H \in \mathbf{RP}^2_\infty$, the S^1-family is just the family of real lines at infinity whose images are real lines in H, (iii) there is a real connected component of the total space of $\tilde{\mathcal{C}}_{U^\sigma} \to U^\sigma$ such that the real circles of each fiber ($\mathbf{CP}^1$) is just the S^1-family of real curves on S_H for any $H \in U^\sigma$, (iv) for any two real planes H and H', and*

for any members $L \subset S_H$ and $L' \subset S_{H'}$ of the S^1-families, L and L' can be connected by deformation in Z preserving the real structure, (v) when a real plane moves to any one of the four planes on which the tropes of B lie, members of the S^1-family converge to the real lines over the tropes.

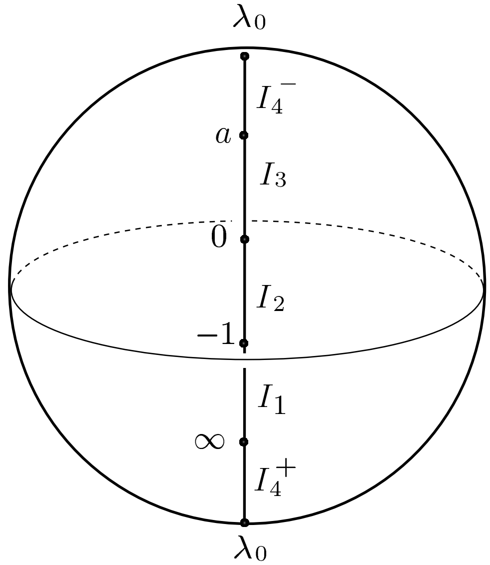

FIGURE 12. $\langle l_\infty \rangle^\sigma$ can be considered as an 'axis' in $\mathbf{U}$.

Proof. This is a compilation of the results we have obtained in this paper so far. In this proof we put $\mathbf{U} = (\mathbf{RP}^3)^\vee \backslash (4 \text{ points})$, where the four points represent the four planes on which the tropes of B are lying. Note that U^σ is an open subset of $\mathbf{U}$. For any $H \in \mathbf{U}$, $S_H = \Phi^{-1}(H)$ is a smooth rational surface with $c_1^2 = 2$ (Proposition 3.5). So we can consider the local system $\mathcal{H}^2$ over $\mathbf{U}$ whose fiber over H is the cohomology group $H^2(S_H, \mathbf{Z})$ ($\simeq \mathbf{Z}^8$). Note that $\mathcal{H}^2 \to \mathbf{U}$ is not necessarily trivial, since $\pi_1(\mathbf{U}) \simeq \pi_1(\mathbf{RP}^3) \simeq \mathbf{Z}_2$. The four intervals I_j, $1 \le j \le 4$, can naturally be considered as a subset of $\mathbf{U}$ (Figure 12). For each I_j consider the S^1-family of real lines on $S_\lambda = S_{H_\lambda}$ for $\lambda \in I_j$ obtained in Proposition 8.5 (ii). As in Section 6, this S^1-family is the real part of a real pencil on S_λ. Let $\mathcal{L}_\lambda \to S_\lambda$ be the associated real line bundle. (In Section 7, $\mathcal{L}_\lambda$ was used to mean the S^1-family of real lines on S_λ. But there should be no confusion since the real lines are simply the real members of $|\mathcal{L}_\lambda|$ for the present $\mathcal{L}_\lambda \to S_\lambda$.) Because these S^1-families form a connected family (Proposition 8.5), the cohomology class determined by $\mathcal{L}_\lambda$ is independent of I_j to which λ belongs. Therefore, by associating the cohomology class of $\mathcal{L}_\lambda$, we obtain a section (of $\mathcal{H}^2$) over $\langle l_\infty \rangle^\sigma \backslash$(the 4 points). This section can be naturally extended to a section over $\mathbf{U} \backslash \mathbf{RP}^2_\infty$, since $\mathbf{U} \backslash \mathbf{RP}^2_\infty$ is contractible to $\langle l_\infty \rangle^\sigma$. On the other hand, we have a section over $\mathbf{RP}^2_\infty$ of $\mathcal{H}^2$, which associates the cohomology class of the line bundle $\Phi^* O_H(1) - \Gamma_0$ for each $H \in \mathbf{RP}^2_\infty$. These two sections fit together to give a global section of $\mathcal{H}^2 \to \mathbf{U}$, since when λ approaches λ_0, $\mathcal{L}_\lambda$ converges to the line bundle $\Phi^* O_H(1) - \Gamma_0$ by Proposition 8.5 (vi). This section must be real, since $\mathcal{L}_\lambda$ is real. Thus we have obtained a real global section of $\mathcal{H}^2 \to \mathbf{U}$. Recalling that S_H is a smooth rational surface (implying $H^2(S_H, \mathbf{Z}) \simeq \operatorname{Pic} S_H$), the section uniquely determines a real line bundle $\mathcal{L}_H \to S_H$ for any $H \in \mathbf{U}$. Now we claim that $|\mathcal{L}_H|$ is a (real) pencil for any $H \in \mathbf{U}$. First, if $H \in \langle l_\infty \rangle^\sigma$ or $H \in \mathbf{RP}^2_\infty$, the claim is obvious from the above construction of the section. (This implies (i) and (ii) of the proposition.) So it suffices to prove the claim for $H \in U^\sigma$. Since we have $H^0(S_\lambda, \mathcal{L}_\lambda) \simeq \mathbf{C}^2$ and $H^i(S_\lambda, \mathcal{L}_\lambda) = 0$ for $i \ge 1$, it is easily verified that $|\mathcal{L}_H|$ is also a (real) pencil for any real plane H sufficiently near to the original planes H_λ. This implies that there exists a

neighborhood W of $\langle l_\infty\rangle^\sigma$ in $\mathbf{U}$ such that $|\mathcal{L}_H|$ is a pencil for any $H \in W$. Real members of this pencil can be assumed to be real lines, since real members of $|\mathcal{L}_\lambda|$ are actually real lines, and since $N \simeq O(1)^{\oplus 2}$ is an open condition. Then by Proposition 3.2, the image of these real lines on S_H must be real touching conics on H and these form a S^1-family of real touching conics on H. If H is moved in U^σ, then the S^1-family of real touching conics automatically survives since as in Proposition 3.10 any smooth quartic has 63 families of touching conics. Taking the inverse image, this implies that $|\mathcal{L}_H|$ is a pencil for any $H \in U^\sigma$, as claimed.

Thus we have specified a real pencil $|\mathcal{L}_H|$ for any $H \in \mathbf{U}$. We claim that any real member of this pencil must be irreducible. For $H \in \langle l_\infty\rangle^\sigma$ this is obvious from our explicit construction of the family of touching conics. For $H \in \mathbf{RP}^2_\infty$, this is also immediate from Proposition 8.1. For $H \in U^\sigma$, $H \cap B$ is a smooth quartic curve. It follows that reducible members of the pencil $|\mathcal{L}_H|$ must be a sum of two smooth rational curves intersecting transversally at a unique point (cf. the proof of Proposition 3.10). Therefore real reducible member of the pencil gives a real point, contradicting the fact that Z has no real point (Proposition 2.5). Thus $|\mathcal{L}_H|$ has no real reducible member for any $H \in \mathbf{U}$.

Then for each $H \in \mathbf{U}$ we associate $|\mathcal{L}_H|^\sigma$ which forms S^1-family of smooth real rational curves on S_H. Then it is obvious from the above construction that (i)–(iv) of the proposition hold. Further, (v) is also immediate from Proposition 8.5 (iv). □

Remark 9.2. In view of our proof of Proposition 9.1, one may expect that the family of line bundles $\mathcal{L}_H \to S_H$, $H \in \mathbf{U}$ are further extended to $\Phi^{-1}(H_i)$, $1 \le i \le 4$, where H_i are the four planes on which the tropes of B lie. However this is not true, since the real line lying over the tropes are not Cartier divisors on $\Phi^{-1}(H_i)$. (Recall that $\Phi^{-1}(H_i)$ is a reducible surface and that the two irreducible components intersect transversally along the real lines over the tropes.)

The following proposition, saying that the real rational curves in Proposition 9.1 foliate Z, is the final main step in our proof of the main theorem given in the next section:

Proposition 9.3. *For any point of Z there uniquely exists a unique real smooth rational curves obtained in Proposition 9.1 going through the point.*

Proof. Because our proof is somewhat long, we give an outline. First we take any real line l in $\mathbf{CP}^3$. Then $E := \Phi^{-1}(l)$ is a real curves in Z which becomes either a smooth elliptic curve or a cycle of smooth rational curves consisting of two or eight irreducible components, depending on how l intersects B. Next we show the following claim:

(♯) For any point of E there uniquely exists a real plane H containing l, on which there exists a member L of the S^1-family (obtained in Proposition 9.1) going through the given point.

This directly implies the claim of the proposition. In the following, let $\mathbf{U}$ and $\mathcal{L}_H \to S_H$ ($H \in \mathbf{U}$) have the same meaning in the previous proof. In particular, the S^1-family of real smooth rational curves on S_H of Proposition 9.1 is just the real part $|\mathcal{L}_H|^\sigma$ of $|\mathcal{L}_H|$. Moreover, for a real line $l \subset \mathbf{CP}^3$, $\langle l\rangle$ denotes the pencil of planes containing l and $\langle l\rangle^\sigma$ denotes its real members parametrized by S^1.

First we consider the case that the real line l intersects B transversally (at four points). In this case $E = \Phi^{-1}(l)$ is a real smooth elliptic curve. For each real plane H containing l we define a subset of E by

$$\mathcal{T}_H = \{L \cap E \mid L \in |\mathcal{L}_H|^\sigma\}.$$

Since $(E, L)_{S_H} = 2$ (by adjunction), and since E has no real point, $L \cap E$ is a pair of conjugate points for any $L \in |\mathcal{L}_H|^\sigma$. It is readily verified that the restriction map $H^0(S_H, \mathcal{L}_H) \to H^0(E, \mathcal{L}_H|_E)$ is isomorphic. Therefore $\mathcal{T}_H$ is also the union of the zero locus of all real

sections of the line bundle $\mathcal{L}_H|_E$. This implies that if $\mathcal{T}_H \cap \mathcal{T}_{H'}$ is non-empty, then $\mathcal{T}_H = \mathcal{T}_{H'}$ (as a subset of E).

Next we investigate structure of $\mathcal{T}_H$. We show in this paragraph that $\mathcal{T}_H$ is a disjoint union of two circles $\mathcal{T}_H^+$ and $\mathcal{T}_H^-$ such that (a) $\sigma(\mathcal{T}_H^+) = \mathcal{T}_H^-$, (b) $\mathcal{T}_H^+$ and $\mathcal{T}_H^-$ are homologous but not zero-homologous. Since the parameter space of $|\mathcal{L}_H|^\sigma$ is a circle, it is obvious from the beginning that $\mathcal{T}_H$ is a (connected) circle or a union of two disjoint circles. To show that the former case cannot happen, consider the double covering $E \to l$. Since E has no real point (because Z has no real point by Proposition 2.5), the four branch points of $E \to l$ are not real. Since l is real, we can decompose l as $l^+ \cup l^\sigma \cup l^-$, where l^σ is the real locus (a circle) of l and $l^\pm$ are hemi-spheres satisfying $\sigma(l^+) = l^-$. First we consider the case that $H \in \langle l\rangle^\sigma$ does not go through P_0. In this case Φ gives an isomorphism between $L \in |\mathcal{L}_H|^\sigma$ and $\Phi(L)$ which is a real touching conic. Hence (because L does not have real point) $\Phi(L) \cap l$ consists of two points, one of which belongs to l^+ and the other belongs to l^-. Therefore the set $\{\Phi(L) \cap l \,|\, L \in |\mathcal{L}_H|^\sigma\} \subset l$ must be two circles. It follows that $\mathcal{T}_H$ is a union of two circles, which we denote by $\mathcal{T}_H^+$ and $\mathcal{T}_H^-$. The decomposition $l = l^+ \cup l^\sigma \cup l^-$ yields the decomposition $E = E^+ \cup \Phi^{-1}(l^\sigma) \cup E^-$, where we set $E^+ = \Phi^{-1}(l^+)$ and $E^- = \Phi^{-1}(l^-)$. We can suppose $\mathcal{T}_H^+ \subset E^+$ and $\mathcal{T}_H^- \subset E^-$. Then obviously $\sigma(\mathcal{T}_H^+) = \mathcal{T}_H^-$ and we get the claim (a) for H not going through P_0. We postpone (b) for planes of this kind and consider a unique real plane $H_0 \in \langle l\rangle^\sigma$ going through P_0. It is readily seen that $\mathcal{T}_{H_0} = \Phi^{-1}(l^\sigma)$. Because the branch points of $E \to l$ are non-real, $\Phi^{-1}(l^\sigma)$ is a circle or a disjoint union of two circles. Suppose the former. Then since $E \backslash \Phi^{-1}(l^\sigma)$ is evidently disconnected, $\Phi^{-1}(l^\sigma)$ becomes homologous to zero and hence bounds a disk. σ clearly has to act on this disk, so has a fixed point by the Brouwer fixed point theorem. This is a contradiction and we conclude that $\Phi^{-1}(l^\sigma)$ is a disjoint union of two circles. We also denote these circles by $\mathcal{T}_{H_0}^+$ and $\mathcal{T}_{H_0}^-$. Then since $\Phi(\mathcal{T}_{H_0}^+) = \Phi(\mathcal{T}_{H_0}^-) = l^\sigma$, we have $\sigma(\mathcal{T}_{H_0}^+) = \mathcal{T}_{H_0}^-$ and we get (a) for $H = H_0$. Further, since σ gives an automorphism of E and since $\mathcal{T}_{H_0}^+$ and $\mathcal{T}_{H_0}^-$ are disjoint, $\mathcal{T}_{H_0}^+$ and $\mathcal{T}_{H_0}^-$ must be mutually homologous. Furthermore, they are not homologous to zero since otherwise $E \backslash \Phi^{-1}(l^\sigma)$ would have three connected components, which is not the case. Thus we obtained (b) for $H = H_0$. This immediately implies (b) for $H \neq H_0$, since when H continuously moves in $\langle l\rangle^\sigma$, $\mathcal{T}_H$ also moves in E continuously.

Next we show that $\{\mathcal{T}_H \,|\, H \in \langle l\rangle^\sigma\}$ gives a foliation of E. Namely we show

$$E = \bigcup_{H \in \langle l\rangle^\sigma} \mathcal{T}_H = \bigcup_{H \in \langle l\rangle^\sigma} (\mathcal{T}_H^+ \cup \mathcal{T}_H^-) \quad \text{(disjoint union)} \tag{67}$$

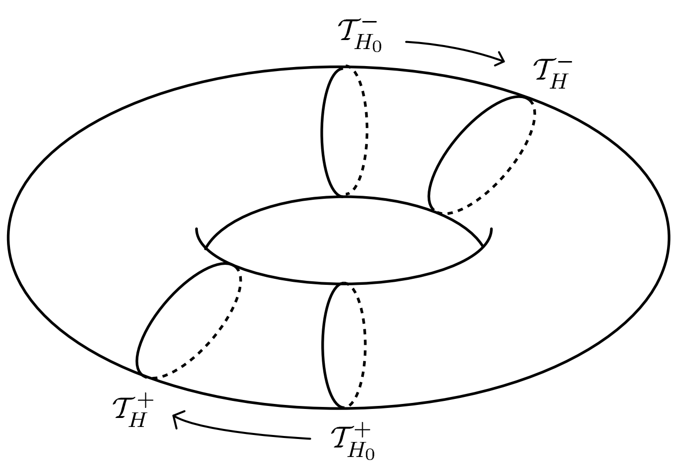

FIGURE 13. foliation on E

(Figure 13) To prove (67) let $\mathrm{Pic}^2 E$ be the set of line bundles on E of degree two. Of course, $\mathrm{Pic}^2 E$ can be identified with $\mathrm{Pic}^0 E$ (though there is no canonical isomorphism). Then we define a map

$$\alpha_l : \langle l\rangle^\sigma \to \mathrm{Pic}^2 E; \quad H \mapsto \mathcal{L}_H|_E.$$

Then $\alpha_l(H) = \alpha_l(H')$ if and only if $\mathcal{T}_H = \mathcal{T}_{H'}$. Further, as is already seen, $\mathcal{T}_H = \mathcal{T}_{H'}$ if and only if they are not disjoint. Hence to prove (67) it suffices to show that α is diffeomorphic. Since the line bundle $\mathcal{L}_H \to S_H$ can be holomorphically extended to that on $S_{H'}$ for any $H' \in \langle l\rangle$ sufficiently near to H, α_l can be extended to a holomorphic map $\tilde{\alpha}_l : A \to \mathrm{Pic}^2 E$, where A is an annulus which is an open neighborhood of $\langle l\rangle^\sigma$ in $\langle l\rangle$. In particular, the ramification index of α makes sense for any point of $\langle l\rangle^\sigma$. Let $H_0 \in \langle l\rangle^\sigma$ be the unique plane satisfying $P_0 \in H_0$ as in the last paragraph. We then have $\alpha_l(H_0) = \Phi^* O_l(2)$. Moreover, since $L \in |\mathcal{L}_H|^\sigma$ is mapped biholomorphically onto a touching conic for any $H \neq H_0$, $\alpha_l(H) \neq \Phi^* O_l(2)$ for any $H \neq H_0$. In particular, α_l is not a constant map. Now recall that we have already seen in Proposition 8.1 that $N_{L/Z} \simeq O(1)^{\oplus 2}$ for $L \in |\mathcal{L}_{H_0}|^\sigma$ and therefore Z has a structure of twistor space in a neighborhood of L. This implies that the ramification index of α_l at H_0 is one, since otherwise L can be deformed in Z in such a way that its Kodaira-Spencer class in $H^0(N_{L/Z})$ vanishes at two points ($= L \cap E$), which means that $N_{L/Z}$ contains $O(2)$ as a subbundle. On the other hand, the image of α_l is contained in $(\mathrm{Pic}^2 E)^\sigma$. Because $\mathrm{Pic}^2 E$ is an elliptic curve, its real locus is, if not empty, either a circle or a union of two circles. In any way, the image of α_l is contained in the connected component of $(\mathrm{Pic}^2 E)^\sigma$ containing $\Phi^* O_l(1)$. In particular the degree of α_l as a differential map from S^1 to S^1 makes sense. Now since we have already shown that the ramification index of α_l at H_0 is one and that $\alpha_l^{-1}(\Phi^* O_l(1)) = \{H_0\}$, it follows that the degree of α_l is one. Hence α_l must be surjective. It remains to show the injectivity of α_l, or more precisely, that the differential $d\alpha_l$ does not vanishes everywhere on $\langle l\rangle^\sigma$. Suppose that $H \in \langle l\rangle^\sigma$ is a critical point of α_l. This implies, as in the proof of Proposition 7.3, that $N_{L/Z}$ contains $O(k)$ as a holomorphic subbundle, where k denotes the ramification index of $\tilde{\alpha}_l$ at $H \in \langle l\rangle$. It is readily seen that $N_{L/Z}$ is either $O \oplus O(2)$ or $O(1)^{\oplus 2}$. Therefore we have $k = 2$ for any critical points (if any). Under the assumption of the existence, the number of critical points of α_l is obviously greater than one. On the other hand, our choice of real lines over $H \in \langle l_\infty\rangle^\sigma$ (done in Section 7.2–7.5) implies that for any $L \in \mathcal{L}_{H_\lambda}$, $H_\lambda \in \langle l_\infty\rangle^\sigma$, we have $N_{L/Z} \simeq O(1)^{\oplus 2}$ (because we have chosen the touching conics on H_λ in such a way that the function h_i $(0 \le i \le 3)$ have no critical points). Therefore, (because $N \simeq O(1)^{\oplus 2}$ is an open condition) we have $N_{L/Z} \simeq O(1)^{\oplus 2}$ for any $L \in |\mathcal{L}_H|^\sigma$, where $H \in \langle l'\rangle^\sigma$, l' being a real line sufficiently near to l_∞. Hence for this real line l', $\alpha_{l'}$ must be diffeomorphic and does not have critical points. Now consider a smooth real one-dimensional family of real lines $\{l_t \,|\, 0 \le t \le 1\}$ such that $l_0 = l'$ and $l_1 = l$, and such that $l_t \cap B$ is transversal for any t. Then the map $\alpha_t = \alpha_{l_t}$ clearly varies differentiably. Because α_0 is diffeomorphic, and because α_1 was assumed to have at least two critical points, there must be some t, $0 < t < 1$, such that α_t has a critical point whose ramification index is greater than 2. This implies that for $L_t \in |\mathcal{L}_{H_t}|^\sigma$, $H_t \in \langle l_t\rangle^\sigma$, $N_{L_t/Z}$ contains $O(k)$, $k \ge 3$ as a subbundle. This is a contradiction and we get that α_l does not have critical points, as claimed. Thus we obtain (67), which of course imply that ($\sharp$) is true for our l intersecting B transversally.

Next we consider the case that the real line l does not intersect B transversally. Then we have $l \not\subset B$ since B does not have real point except P_0. Hence $l \cap B$ consists of one, two or three points. If $l \cap B$ is one or three point, then it must contain P_0 by reality. But we have already shown in the proof of Proposition 8.1 that if a real line l goes through P_0, then $l \cap B$ consists of three points. Hence we consider the case that $l \cap B$ consists of three points, one of which is P_0. In this case $\Phi^{-1}(l)$ is a union of the exceptional curve Γ_0 and the real line L over l, intersecting transversally at two points (cf. the proof of Proposition 8.1). As proved in Proposition 8.1, real lines L of this form (which were called 'at infinity') are

disjoint. Furthermore, the isomorphism ψ between $\Gamma_0/\langle\sigma\rangle$ and the projective space of real lines through P_0 in the proof means that, for any point of Γ_0, there uniquely exists a real line at infinity through the point. This directly implies ($\sharp$) for real lines l going through P_0.

So it remains to see the case that $l \cap B$ consists of a conjugate pair of points; namely l being a real bitangent. By Proposition 3.14, such a bitangent l must be always trivial, which means that l lies on one of the four real planes on which the tropes lie. Let $H_i \supset l$ denote this real plane. First we assume $l \neq l_\infty$. In this case $\Phi^{-1}(l)$ becomes a cycle of two smooth rational curves intersecting transversally over the tangent points. We write $\Phi^{-1}(l) = E = E^+ + E^-$ with $E^- = \overline{E}^+$, and P_i and $\overline{P}_i$ denote the intersection points of E^+ and E^-. Further, write $\Phi^{-1}(H_i) = D_i + \overline{D}_i$ and $L_i = D_i \cap \overline{D}_i$, which is the only candidate of twistor line lying on $\Phi^{-1}(H_i) = D_i + \overline{D}_i$. Then we have $L_i \cap E = \{P_i, \overline{P}_i\}$. Next, for any real plane $H \supset l$ different from H_i, we set $\mathcal{T}_H = \{L \cap E \,|\, L \in |\mathcal{L}_H|^\sigma\}$. Then as in the case of E being a smooth elliptic curve, each $L \cap E$ consists of a conjugate pair of points and $\mathcal{T}_H$ is a disjoint union of two circles $\mathcal{T}_H^+$ and $\mathcal{T}_H^-$, one of which is contained in E^+ and the other is in E^-. We claim that

$$E\backslash\{P_i, \overline{P}_i\} = \bigcup_{H \in \langle l\rangle^\sigma \backslash \{H_i\}} \mathcal{T}_H \tag{68}$$

holds; namely $E\backslash\{P_i, \overline{P}_i\}$ is foliated by $\mathcal{T}_H$ (Figure 14).

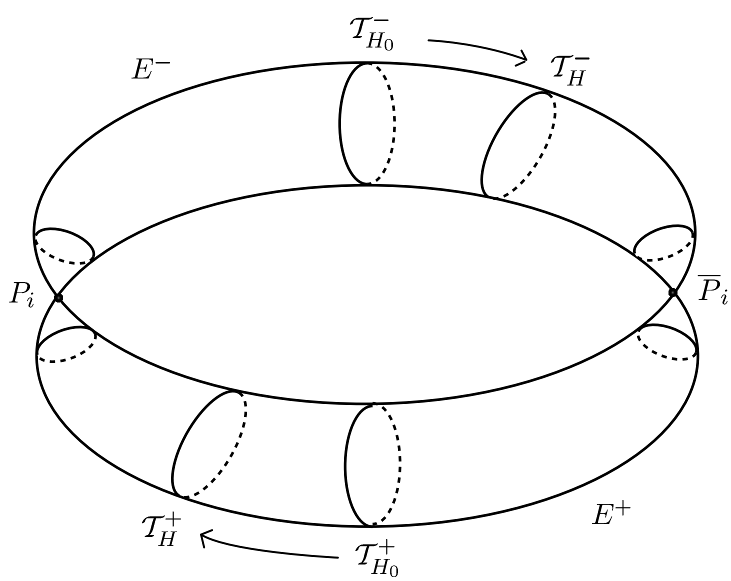

FIGURE 14. foliation on $E = E^+ + E^-$

To show this, let $\mathrm{Pic}^{1,1}E$ denote the subset of the Picard group of E whose elements represent holomorphic line bundles on E whose degree of their restriction onto E^+ and E^- are both one. Take an affine coordinate z (resp. w) on E^+ (resp. E^-), whose origin represents P_i and infinity represents $\overline{P}_i$.

Claim. Suppose that $F \in \mathrm{Pic}^{1,1}E$ is real (i.e. $\sigma^* F \simeq \overline{F}$). Then the union of the zero locus of real sections of F consists of two circles $\mathcal{T}^+ \subset E^+$ and $\mathcal{T}^- \subset E^-$ having the radiuses $|z| = r > 0$ and $|w| = 1/r$ respectively. Moreover, $F \simeq F'$ iff the two circles coincides respectively.

We postpone a proof of this claim and continue the proof of Proposition 9.3. We define a map

$$\alpha_l : \langle l\rangle^\sigma\backslash\{H_i\} \to \ \mathrm{Pic}^{1,1}E; \quad H \mapsto \mathcal{L}_H|_E.$$

Then by definition $\mathcal{T}_H$ is again the union of the zero locus of real sections of $\alpha_l(H) = \mathcal{L}_H|_E$. Since $L \cap E$ belongs to the linear system $|\mathcal{L}_H|_E|$ for any $L \in |\mathcal{L}_H|^\sigma$, $\mathcal{T}_H = \mathcal{T}_{H'}$ holds iff $\mathcal{T}_H$ and $\mathcal{T}_{H'}$ intersect. Thus by the above claim, to prove (68), it suffices to show that α_l is a bijection onto $(\mathrm{Pic}^{1,1}E)^\sigma$. First we show the surjectivity. We have seen in Proposition 9.1 (v) that as the real plane H goes to H_i, members of the S^1-family $|\mathcal{L}_H|^\sigma$ converge to L_i. Hence the two circles $\mathcal{T}_H^+$ and $\mathcal{T}_H^-$ will shrink to the two points P_i and $\overline{P}_i$ as H goes to H_i. On the other hand, we have $N_{L_i/Z} \simeq O(1)^{\oplus 2}$ (Lemma 7.10). Therefore Z has a structure of twistor space in a neighborhood of L_i. In particular, all small deformations of L_i in Z keeping the reality must be disjoint. This implies that, when a real plane H moves in $\langle l\rangle^\sigma$ to pass H_i from one side to another side, then the intersection circle $\mathcal{T}_H^+$ and $\mathcal{T}_H^-$ moves to pass (via shrinking) from E^+ to E^- (or E^- to E^+) without turning the direction. This implies that $\mathcal{T}_H$ sweeps the whole of $E\backslash\{P_i, \overline{P}_i\}$ when H runs through $\langle l\rangle^\sigma\backslash\{H_i\}$. Hence α_l must be surjective onto $(\mathrm{Pic}^{1,1}E)^\sigma$. The injectivity can be proved in a similar way as in the case that $l \cap B$ is transversal. Therefore α_l must be injective and (68) holds. Hence ($\sharp$) is proved for a real bitangent $l \neq l_\infty$.

Finally, we prove that ($\sharp$) is true for $l = l_\infty$, or equivalently,

$$E\backslash\{P_i, \overline{P}_i\}_{i=1}^4 = \bigcup_{H \in \langle l_\infty\rangle^\sigma \backslash \{H_i\}_{i=1}^4} \mathcal{T}_H, \tag{69}$$

where $E = \Phi^{-1}(l_\infty)$, $\mathcal{T}_H$ have the same meaning as before, and $\{P_i, \overline{P}_i\}_{i=1}^4$ denote the singular points of E. This is possible thanks to the explicit construction of real lines on S_λ given in Sections 5–7. By Proposition 2.10, (see Figure 8), $\Phi^{-1}(l_\infty)$ is a cycle of eight rational curves. First consider the case when we take the small resolution determined by (I) of Proposition 7.24. If H is of the form $H_\lambda = \{y_0 = \lambda y_1\}$ where $\lambda \in I_1$ (resp. $\lambda \in I_3$), then L_θ^+ (resp. L_θ^-) must be chosen by Proposition 7.25 (i), where $L_\theta^\pm$ is of course the irreducible components of the inverse images of touching conics of special type. Hence by Lemma 7.14, $\mathcal{T}_H$ is two circles, one of which is contained in Γ_1 (resp. Γ_3) and the other is contained in $\overline{\Gamma}_1$ (resp. $\overline{\Gamma}_3$). Further, their radiuses are given by the function h_1 (resp. h_3) defined after Lemma 7.14. For our choice of $\ell_1(= x_1)$, Lemma 7.16 (i) implies that the function h_1 gives a bijection between I_1 and $\mathbf{R}^+$. This implies that $\mathcal{T}_{H_\lambda}$ foliates $\Gamma_1 \cup \overline{\Gamma}_1$ (with the $U(1)$-fixed points removed) when λ runs through I_1. In the same manner, $\mathcal{T}_{H_\lambda}$ foliates $\Gamma_3 \cup \overline{\Gamma}_3$ (with the $U(1)$-fixed points removed) when λ runs through I_3. Next, if H is of the form $H_\lambda = \{y_0 = \lambda y_1\}$ where λ is in I_2 then by Lemma 7.19, $\mathcal{T}_H$ is two circles, one of which is contained in Γ_2 and the other is contained in $\overline{\Gamma}_2$. Moreover, their radiuses are given by the function h_2 defined in Proposition 7.21. For our choices of $\ell_1(= x_1)$ and $\ell_2(= x_0 + x_1)$, Lemma 7.22 (iii) implies that h_2 gives a bijection between I_2 and $\mathbf{R}^+$. This implies that $\mathcal{T}_{H_\lambda}$ foliates $\Gamma_2 \cup \overline{\Gamma}_2$ (with the two endpoints removed) when λ runs through I_2. Finally when H is of the form $H_\lambda = \{y_0 = \lambda y_1\}$ where λ is in $I_4^- \cup I_4^+$, then $\mathcal{T}_H$ is two circles, one of which is contained in Ξ and the other is contained in $\overline{\Xi}$. Further, their radiuses are given by h_0 and h_0^{-1}. Thus if we recall that L_θ^+ has to be chosen (Proposition 7.25), and if we recall the rule of distinction of L_θ^+ and L_θ^- made in the explanation just before Proposition 7.25, $\mathcal{T}_{H_\lambda}$ covers $\Xi \cup \overline{\Xi}$ with the unit circles in Ξ and $\overline{\Xi}$, and the four $U(1)$-fixed points removed. The unit circles (in Ξ and $\overline{\Xi}$) are covered by real lines in H_{λ_0} whose images are real lines going through P_0. Thus $\mathcal{T}_{H_\lambda}$ foliates $l \cup \bar{l}$ (with the four $U(1)$-fixed points removed) when λ runs through I_4. Finally, the remaining points, that is, the eight singular points of E are passed by L_i, $1 \leq i \leq 4$. Thus we have verified that $\mathcal{T}_H$ foliate $\Phi^{-1}(l_\infty)$ as H runs through $\langle l_\infty\rangle^\sigma$, and we have completed the claim of Proposition 9.3. □

Proof of the claim. Let $F \in \mathrm{Pic}^{1,1}E$ be a line bundle which is not necessarily real. Because $F|_{E^+} \simeq O(1)$ and $F|_{E^-} \simeq O(1)$, F must be the line bundle associated to the effective divisor $a + b$ on E, where $a \in \mathbf{C}^*$ (resp. $b \in \mathbf{C}^*$) represents a point on $E^+\backslash\{P_i, \overline{P}_i\}$

(resp. $E^-\backslash\{P_i,\overline{P}_i\}$) satisfying $z=a$ (resp. $w=b$). Namely, we have $F=[a+b]$ in the usual notation. Then it is elementary to see that for $(a',b')\in\mathbf{C}^*\times\mathbf{C}^*$ the divisor $a'+b'$ is linearly equivalent to $a+b$ iff $(a',b')=(ca,cb)$ for some $c\in\mathbf{C}^*$. Indeed, the condition of linear equivalence is the same thing as the existence of a meromorphic function f on E having $z=a$ and $w=b$ as simple zeros, and $z=a'$ and $w=b'$ as simple poles. This f must be of the form $c(z-a)/(z-a')$ on E^+ for some $c\in\mathbf{C}^*$ and $c'(w-b)/(w-b')$ on E^- for some $c'\in\mathbf{C}^*$. Further, the values of these two functions must coincide at P_i (i.e. $z=w=0$) and $\overline{P}_i$(i.e. $z=w=\infty$). This implies that $a/b=a'/b'$. Now consider a map from $\mathrm{Div}^{1,1}_+E$ to $\mathbf{C}^*$ sending $a+b$ to a/b, where $\mathrm{Div}^{1,1}_+E$ denotes the set of effective Cartier divisors whose degrees on $E^{\pm}$ are one. Then the above argument implies that this map descends to give an isomorphism $\mathrm{Pic}^{1,1}E\simeq\mathbf{C}^*$.

Next assume that $F\in\mathrm{Pic}^{1,1}E$ is real and s is a real section of F. Recall that there is a (branched) double covering $E\to l$ commuting with the real structures, where the real structure on E interchanges E^+ and E^-, and the real structure on l (a real line) is a complex conjugation. Therefore we can write $F=[a+1/\overline{a}]$ for some $a\in\mathbf{C}^*$ Thus the image of F in $\mathbf{C}^*$ by the above isomorphism $\mathrm{Pic}^{1,1}E\to\mathbf{C}^*$ is $(1/\overline{a})/a=1/|a|^2$. This implies the image of real line bundles is $\mathbf{R}_{>0}$. (Although not needed in the sequel, this calculation also implies that if we change the real structure on l from the complex conjugation to the antipodal map (and yet assuming $\sigma(E^+)=E^-$ and that $E\to l$ commutes with the real structures), then F becomes real iff the image is in $\mathbf{R}_{<0}$.) It is immediate to see that for a real divisor $a+1/\overline{a}\in\mathrm{Div}^{1,1}_+E$, $\alpha a+(\alpha/\overline{a})$ remains to be real iff $\alpha\in U(1)$. This implies that the union of the zero locus of a real line bundle $F=[a+1/\overline{a}]$ is two circles $|z|=|a|$ (in E^+) and $|w|=1/|a|$ (in E^-), which implies the claim. □

10. The main theorems

In this section, based on the results in the previous section, we prove our main result that our quartic surfaces always determine twistor spaces of $3\mathbf{CP}^2$ whose self-dual metrics are non-LeBrun while admitting a non-trivial Killing field (Theorem 10.1). Then by using this, we determine the moduli space of these self-dual metrics on $3\mathbf{CP}^2$ (Theorem 10.4).

In order to state our result precisely, we now recall the setting. Let (y_0,y_1,y_2,y_3) be a homogeneous coordinate on $\mathbf{CP}^3$ and consider a real structure defined by

$$(y_0,y_1,y_2,y_3)\mapsto(\overline{y}_0,\overline{y}_1,\overline{y}_3,\overline{y}_2). \tag{70}$$

Let B be a quartic surface defined by

$$(y_2y_3+Q(y_0,y_1))^2-y_0y_1(y_0+y_1)(y_0-ay_1)=0, \tag{71}$$

where a is a positive real numbers, and Q is a quadratic form of y_0 and y_1 with real coefficients. Then B is preserved by the real structure. Suppose that Q and a satisfy the following properties which are necessary conditions for getting a twistor space (Proposition 2.6):

Condition (A): *If λ satisfies $\lambda(\lambda+1)(\lambda-a)\ge 0$ (i.e. if $\lambda\in I_2\cup I_4$), then*

$$Q(\lambda,1)\ge\sqrt{\lambda(\lambda+1)(\lambda-a)} \tag{72}$$

holds. Moreover, there exists a unique λ_0, $\lambda_0>a$ (i.e. $\lambda_0\in I_4$) such that the equality of (72) holds for $\lambda=\lambda_0$.

(We can assume $\lambda_0>a$ by possible application of a projective transformation $(y_0,y_1)\mapsto(ay_1,-y_0)$ as explained in Section 7.1. This assumption will be important afterwards.) Then B has three singular points P_0, P_∞ and $\overline{P}_\infty$, where P_0 is a real ordinary double point and $\{P_\infty,\overline{P}_\infty\}$ is a pair of conjugate points which are simple elliptic singularities of type $\tilde{E}_7$ (Proposition 2.2). Let $Z_0\to\mathbf{CP}^3$ be the double covering branched along B. Since B is

real, the real structure on $\mathbf{CP}^3$ naturally induces that on Z_0. Z_0 has three singular points p_0, p_∞ and $\overline{p}_\infty$ over the singularities of B respectively. Then the following result, saying in particular that a quartic surface (71) canonically determines (via twistor space) a self-dual metric on $3\mathbf{CP}^2$, is the main result of this paper:

Theorem 10.1. *(i) There are precisely two small resolutions of Z_0 such that the resulting compact complex threefolds are twistor spaces of $3\mathbf{CP}^2$. (ii) The two self-dual metrics on $3\mathbf{CP}^2$ defined by the two twistor spaces are mutually conformally isometric. (iii) The self-dual metrics on $3\mathbf{CP}^2$ determined by the above twistor spaces have an isometric $U(1)$-action but are not conformally isometric to LeBrun metric. (iv) For any non-LeBrun self-dual metric on $3\mathbf{CP}^2$ of positive scalar curvature admitting an isometric $U(1)$-action, there exists a quartic surface (71) satisfying Condition (A) such that the resulting twistor space is the twistor space of the given self-dual metric.*

(Concerning (iv) of the theorem, there are two choices of quartic surface representing given self-dual metric in general, as will be shown in Lemma 10.3.)

Proof of Theorem 10.1. By Propositions 7.24 and 8.5 there are precisely two small resolutions of Z_0 such that the resulting non-singular threefolds Z can have a structure of twistor space. We show that these two threefolds are actually twistor spaces of $3\mathbf{CP}^2$. By Proposition 9.1, Z has a connected family of real smooth rational curves. Moreover, by Proposition 9.3, different members of this family are disjoint and Z is foliated by the curves of this family. Therefore Z must be a twistor space of some four-dimensional manifold M, having this family as the set of twistor lines. By Lemma 7.9 or 7.10, Z always possesses a divisor $D+\overline{D}$ such that D is biholomorphic to a three points blown-up of $\mathbf{CP}^2$. In particular, D is diffeomorphic to $\mathbf{CP}^2\#(\overline{\mathbf{CP}}^2\#\overline{\mathbf{CP}}^2\#\overline{\mathbf{CP}}^2)$, where $\overline{\mathbf{CP}}^2$ denotes the complex projective plane with the complex orientation reversed. Then $L=D\cap\overline{D}$ is a real line in Z (Lemma 7.10) and L is a member of our family of real lines in Z (Proposition 9.1 (v)). Since D and $\overline{D}$ intersect transversally along L, we have $(N_{D/Z})|_L\simeq N_{L/\overline{D}}$. Hence $(D,L)_Z=\deg([D]|_L)=\deg(N_{D/Z}|_L)=\deg(N_{L/\overline{D}})$, and the last term is one. Thus we get $(D,L)_Z=1$. It follows that $(D,L')_Z=1$ for any member L' of our family. Moreover, our construction of the family implies that L is the unique member which is contained in $D+\overline{D}$. Therefore for any $L'\neq L$, L' intersects D (and $\overline{D}$) transversally at a unique point. This implies that the parameter space of our family can be identified with a one-point compactification of $D\backslash L$. It is easily seen that the latter space is diffeomorphic to $3\mathbf{CP}^2$ with the usual complex orientation reversed. Thus we have shown that Z is a twistor space of $3\mathbf{CP}^2$ and we get (i) of the theorem.

Next we show (ii). We remark that our family of real lines is a unique family which gives Z a structure of twistor space, since by Proposition 3.2 real lines in Z intersecting Γ_0 must be of the form $\Phi^{-1}(l)-\Gamma_0$ (which is actually contained in our family of real lines), where l is a real line through P_0. Hence to prove (ii) it suffices to construct a biholomorphic map ϕ between the two twistor spaces, such that ϕ commutes with the real structures on the twistor spaces, because we have the above uniqueness of the family of twistor lines. Let $\iota:\mathbf{CP}^3\to\mathbf{CP}^3$ be a holomorphic involution defined by $\iota(y_0,y_1,y_2,y_3)=(y_0,y_1,y_3,y_2)$. It is immediate to verify that ι commutes with our real structure on $\mathbf{CP}^3$, ι preserves B, and that $\iota(P_0)=P_0,\iota(P_\infty)=\overline{P}_\infty$ hold. Moreover, ι naturally lifts on the line bundle $O(2)\to\mathbf{CP}^3$ in such a way that $\iota^*x_0=x_0',\iota^*x_1=x_1',\iota^*x_2=x_3'$ and $\iota^*z=z$ hold, where $(x_0,x_1,x_2)=(y_0/y_3,y_1/y_3,y_2/y_3)$ and $(x_0',x_1',x_3')=(y_0/y_2,y_1/y_2,y_3/y_2)$ are non-homogeneous coordinates on $y_3\neq0$ and $y_2\neq0$ respectively as in the proof of Lemma 7.9, and z is a fiber coordinate on the bundle $O(2)\to\mathbf{CP}^3$. If we still denote ι for this lift on $O(2)$, ι preserves $Z_0\subset O(2)$ invariant, and ι commutes with the real structure on $O(2)$. Moreover, by (59), ι interchanges two factors of $\mathbf{CP}^1\times\mathbf{CP}^1$ that is the projectified

tangent cone of Z_0 at P_0. Hence ι is lifted to give a a biholomorphic map between two small resolutions of Z_0, and we get the desired biholomorphic map ϕ. Thus we obtain (ii).

To prove (iii) let ρ be a $U(1)$-action on $\mathbf{CP}^3$ defined by $(y_0, y_1, y_2, y_3) \mapsto (y_0, y_1, e^{i\theta}y_2, e^{-i\theta}y_3)$, $e^{i\theta} \in U(1)$. It is immediate to see that ρ commutes with the real structure on $\mathbf{CP}^3$ and preserves B. Hence it is naturally lifted on Z_0 to be a (holomorphic) $U(1)$-action commuting with the real structure. Moreover, since ρ fixes the singular points of B, this $U(1)$-action on Z_0 is automatically lifted on a small resolution Z. This action on Z is clearly a holomorphic $U(1)$-action commuting with the real structure. Therefore it induces an isometric $U(1)$-action of the corresponding self-dual metric on $3\mathbf{CP}^2$. Moreover, the self-dual metrics are not conformally isometric to LeBrun metrics, since $|(-1/2)K_Z|$ of our twistor spaces do not have base point (cf. Proposition 2.8) but LeBrun twistor spaces have. Thus we get (iii).

Finally, (iv) is obvious from Proposition 2.1. $\square$

We use Theorem 10.1 to determine the moduli space of self-dual metrics on $3\mathbf{CP}^3$ of this kind. We begin with the following

Lemma 10.2. *Let $\tilde{\mathcal{M}}$ be the set of the quartic surface (71) satisfying Condition (A). Then $\tilde{\mathcal{M}}$ is diffeomorphic to $\mathbf{R}^3$.*

Proof. Since the quartic (71) is determined by a quadratic form $Q(y_0, y_1)$ and $a > 0$, we represent a point of $\tilde{\mathcal{M}}$ by a pair (a, Q). Let $\mathcal{N}$ be an open subset of $\mathbf{R}^2$ defined by

$$\mathcal{N} = \{(x, y) \in \mathbf{R}^2 \mid x > 0,\ y > x\}.$$

We consider the map $\varpi : \tilde{\mathcal{M}} \to \mathcal{N}$ defined by $\varpi(a, Q) = (a, \lambda_0)$, where λ_0 is the unique real number attaining the equality of (72). Then it is obvious that ϖ is surjective. Fix $(a, \lambda_0) \in \mathcal{N}$ and set $f(\lambda) = \lambda(\lambda + 1)(\lambda - a)$ as before.

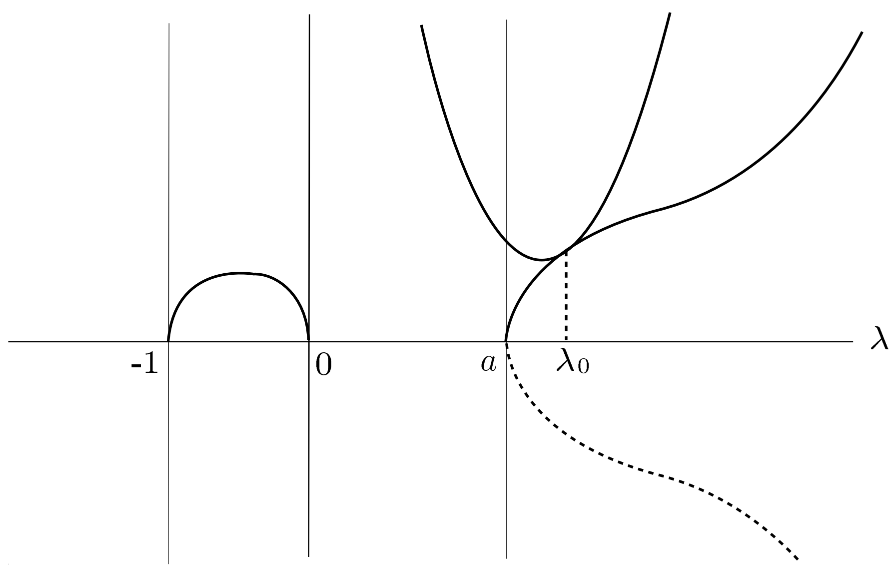

FIGURE 15. graphs of $\sqrt{f}$ and Q

The graph of $\sqrt{f(\lambda)}$ is as in Figure 15 and $\varpi^{-1}(a, \lambda_0)$ is the set of $Q(y_0, y_1)$ such that the graph of $Q(\lambda, 1)$ is over that of $\sqrt{f}$ (on $I_2 \cup I_4$) and is tangent (from above) at the point $(\lambda_0, \sqrt{f(\lambda_0)})$. Writing $Q(y_0, y_1) = by_0^2 + cy_0y_1 + dy_1^2$, the last condition is equivalent to the two equalities

$$b\lambda_0^2 + c\lambda_0 + d = \sqrt{f(\lambda_0)}, \quad 2b\lambda_0 + c = \left(\sqrt{f(\lambda)}\right)'\Big|_{\lambda=\lambda_0}. \tag{73}$$

These define a line in $\mathbf{R}^3$. Moreover, an elementary argument shows that there is a positive constant $b_0 = b_0(\lambda_0) > 0$ such that Condition (A) (for fixed λ_0) is equivalent to (73) together with the inequality $b > b_0(\lambda)$. Thus we get that $\varpi^{-1}(a, \lambda_0)$ is a half-line in $\mathbf{R}^3$. Namely $\varpi : \tilde{\mathcal{M}} \to \mathcal{N}$ is a half-line bundle on $\mathcal{N}$. On the other hand it is readily seen that $\mathcal{N}$ is diffeomorphic to $\mathbf{R}^2$. Hence $\tilde{\mathcal{M}}$ is the total space of $\mathbf{R}$-bundle over $\mathbf{R}^2$. Therefore $\tilde{\mathcal{M}}$ must be diffeomorphic to $\mathbf{R}^3$, as desired. $\square$

At first sight $\tilde{\mathcal{M}}$ may be thought of a moduli space of self-dual metrics. However, there is a possibility that different quartics of the form (71) determines the same (i.e. conformally isometric) self-dual metric. Actually, we have the following:

Lemma 10.3. *Let (a, Q) and (a', Q') be points of $\tilde{\mathcal{M}}$, and B and B' the associated quartic surfaces defined by (71) respectively. Let g and g' be the self-dual metrics on $3\mathbf{CP}^2$ canonically associated to B and B' respectively by Theorem 10.1. Then g and g' are conformally isometric if and only if the following two conditions are satisfied. (i) $a = a'$, (ii) $Q'(ay_0 + ay_1, y_0 - ay_1) = a(a+1)Q(y_0, y_1)$.*

Proof. If there is a conformal isometry between g and g', it naturally defines a biholomorphic map between the twistor spaces. This automorphism also defines an automorphism G of $\mathbf{CP}^3$ commuting with the real structure (70), because $\mathbf{CP}^3$ is canonically identified with the dual projective space of $H^0(Z, (-1/2)K_Z) \simeq \mathbf{C}^4$, and the automorphism of the twistor space naturally lifts on the line bundle $(-1/2)K_Z$. We have clearly $G(B) = B'$. So suppose that G is an automorphism of $\mathbf{CP}^3$ commuting with the real structure (70), and satisfies $G(B) = B'$. Of course, G is a projective transformation.

Assume first that $G(P_\infty) = P_\infty$. Then it follows $G(\overline{P}_\infty) = \overline{P}_\infty$ and it can be easily verified that G must be of the form

$$G = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}, \quad A_1 \in GL(2, \mathbf{R}), \quad A_2 = \begin{pmatrix} a_2 & 0 \\ 0 & \overline{a}_2 \end{pmatrix}, \quad a_2 \neq 0. \tag{74}$$

Moreover, we have $G(B) = B'$ if and only if the following two conditions are satisfied:

$$Q'(y_0', y_1') = |a_2|^2 Q(y_0, y_1) \tag{75}$$

and

$$y_0' y_1' (y_0' + y_1')(y_0' - a' y_1') = |a_2|^4 y_0 y_1 (y_0 + y_1)(y_0 - a y_1), \tag{76}$$

where we put ${}^t(y_0', y_1') = A_1 \cdot {}^t(y_0, y_1)$. Because the cross-ratio is invariant under projective transformations, (76) implies $a = a'$. It is elementary to show that if $a \neq 1$, there are just three linear transformations of $\mathbf{CP}^1$ preserving the four points $\{-1, 0, a, \infty\}$, where we put $-1 = (-1, 1), 0 = (0, 1), a = (a, 1)$ and $\infty = (1, 0)$ and that they are concretely given by

$$A_1^{(1)} = \begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}, \quad A_1^{(2)} = \begin{pmatrix} a & a \\ 1 & -a \end{pmatrix}, \quad A_1^{(3)} = \begin{pmatrix} 1 & -a \\ -1 & -1 \end{pmatrix}.$$

Note that these define involutions on $\mathbf{CP}^1$, $A_1^{(1)}$ preserves orientation of $\mathbf{RP}^1 \simeq S^1$, $A_1^{(2)}$ and $A_1^{(3)}$ reverse it, $A_1^{(1)}$ and $A_1^{(3)}$ interchange $I_2 = (-1, 0)$ and $I_4 = (a, \infty)$, $A_1^{(2)}$ preserves I_2 and I_4, and that $A_1^{(1)} = A_1^{(2)} A_1^{(3)}$ in $PGL(2, \mathbf{R})$. (Thus $A_1^{(1)}, A_1^{(2)}$ and $A_1^{(3)}$ generate a subgroup of $PGL(2, \mathbf{R})$ which is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.) Thus $A_1^{(1)}$ and $A_1^{(3)}$ do not preserves Condition (A) (since they interchange I_2 and I_4 and hence does not preserve the property $\lambda_0 \in I_4$). Therefore, at least if $a \neq 1$, $A_1^{(2)}$ is the unique one which can determine G satisfying $G(B) = B'$. It is immediate to check that under $A_1^{(2)}$, the right hand side of (76) is just multiplied by $a^2(a+1)^2$. Hence we obtain $|a_2|^2 = a(a+1)$. Substituting this into (75), we get $Q'(y_0', y_1') = a(a+1)Q(y_0, y_1)$. This is the equation in (ii) of the lemma. If $a = 1$, there arise other A_1's preserving the four points. But all of them satisfies

$y_0'y_1'(y_0'+y_1')(y_0'-y_1') = cy_0y_1(y_0+y_1)(y_0-y_1)$ with $c<0$ (or more precisely $c=-1$ or -4), and is not compatible with (75). Thus even if $a=1$, we have $A_1 = A_1^{(2)}$.

Next, assume that $G(P_\infty) = \overline{P}_\infty$. Then it can be also easily seen that

$$G = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}, \quad A_1 \in GL(2,\mathbf{R}), \quad A_2 = \begin{pmatrix} 0 & a_2 \\ \overline{a}_2 & 0 \end{pmatrix}, \quad a_2 \neq 0. \tag{77}$$

Once this is obtained, it is completely parallel to the last paragraph to conclude that $a=a'$, $A_1 = A_1^{(2)}$ and $|a_2|^2 = a(a+1)$. Hence we again get $Q'(ay_0+ay_1, y_0-ay_1) = a(a+1)Q(y_0,y_1)$ and we have proved the necessity.

Conversely, suppose that (a,Q) and $(a',Q') \in \tilde{\mathcal{M}}$ satisfy $a=a'$ and $Q'(ay_0+ay_1, y_0-ay_1) = a(a+1)Q(y_0,y_1)$. Then for the real projective transformation

$$G : (y_0,y_1,y_2,y_3) \mapsto \left(ay_0+ay_1, y_0-ay_1, \sqrt{a(a+1)}y_2, \sqrt{a(a+1)}y_3\right),$$

we readily have $G(B) = B'$. This G (automatically) maps singular points of B onto B'. Therefore G naturally lifts on a small resolution of the double covers. Thus G gives rise to an isomorphism of the two twistor spaces. Hence the converse is also proved. □

We use these two lemmas to obtain the following result which describes a global structure of the moduli space:

Theorem 10.4. *Let $\mathcal{M}$ be the set of conformal classes of self-dual metrics g on $3\mathbf{CP}^2$ satisfying the following properties: (i) the scalar curvature of g is positive, (ii) g admits a non-trivial Killing field, (iii) g is not conformally isometric to LeBrun's explicit self-dual metrics* [10]. *Then $\mathcal{M}$ is naturally identified with an orbifold $\mathbf{R}^3/G$, where G is an involution of $\mathbf{R}^3$ having two-dimensional fixed locus.*

Proof. By Lemma 10.3, $\mathcal{M}$ is naturally identified with $\tilde{\mathcal{M}}/G$, where G is an involution on $\tilde{\mathcal{M}}$ defined by

$$(a, Q(y_0,y_1)) \mapsto \left(a, \ \frac{1}{a(a+1)}Q(ay_0+ay_1, y_0-ay_1)\right). \tag{78}$$

By Lemma 10.2, $\tilde{\mathcal{M}}$ is diffeomorphic to $\mathbf{R}^3$. Therefore, in order to prove the theorem, it suffices to show that G acts on $\tilde{\mathcal{M}} \simeq \mathbf{R}^3$ having two-dimensional fixed locus. To see this, write $Q(y_0,y_1) = by_0^2 + cy_0y_1 + dy_1^2$ with $b,c,d \in \mathbf{R}$, as in the proof of Lemma 10.2. Then as in the proof, we can use (a,λ_0,b) as a coordinate on $\tilde{\mathcal{M}}$, because c and d are uniquely determined from b by (73). Write $G(a,\lambda_0,b) = (a',\lambda_0',b')$. Then (78) means that

$$a' = a, \quad \lambda_0' = \frac{a\lambda_0 + a}{\lambda_0 - a}. \tag{79}$$

Moreover, (78) also implies that b' is the coefficient of y_0^2 of the polynomial

$$\frac{1}{a(a+1)}Q(ay_0+ay_1, y_0-ay_1). \tag{80}$$

Now we assert that the fixed locus of G is precisely the set $\{(a,\lambda_0,b) \in \tilde{\mathcal{M}} \,|\, \lambda_0 = a + \sqrt{a^2+a}\}$. This directly implies the theorem. To show the assertion, first it is obvious that λ_0 must be the fixed point of the fractional transformation in (79), which is easily known to be $a+\sqrt{a^2+a}$. On the other hand, by (80), we have $b' = (ba^2+ca+d)/a(a+1)$, where c and d are determined by (73). Substituting $\lambda_0 = a+\sqrt{a^2+a}$, various cancellation occurs and we finally get the last one becomes just b. Therefore, if $\lambda_0 = a+\sqrt{a^2+a}$, then b gets no effect from G. This proves the assertion and we obtain the theorem.

□

By the above proof, a self-dual metric on $3\mathbf{CP}^2$ of positive scalar curvature with a non-trivial Killing field admits an isometry not contained in the $U(1)$-isometry generated by the Killing field, iff the corresponding point of the moduli space $\mathcal{M}$ is a G-fixed point. Moreover, in such a case, the isometry must be an involution.

We also note that by construction of LeBrun [10] and his classification theorem [11] the moduli space of self-dual metrics on $3\mathbf{CP}^2$ whose isometry group is just $U(1)$ (i.e. cannot extends to $U(1)^2$-isometry) acting semi-freely on $3\mathbf{CP}^2$ is identified with the set of different non-collinear three points on the hyperbolic three-space modulo the isometry group. In particular, it has dimension $3 \times 3 - 6 = 3$. It will be really interesting to try to find an explicit description of the self-dual metrics associated to our quartics.

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